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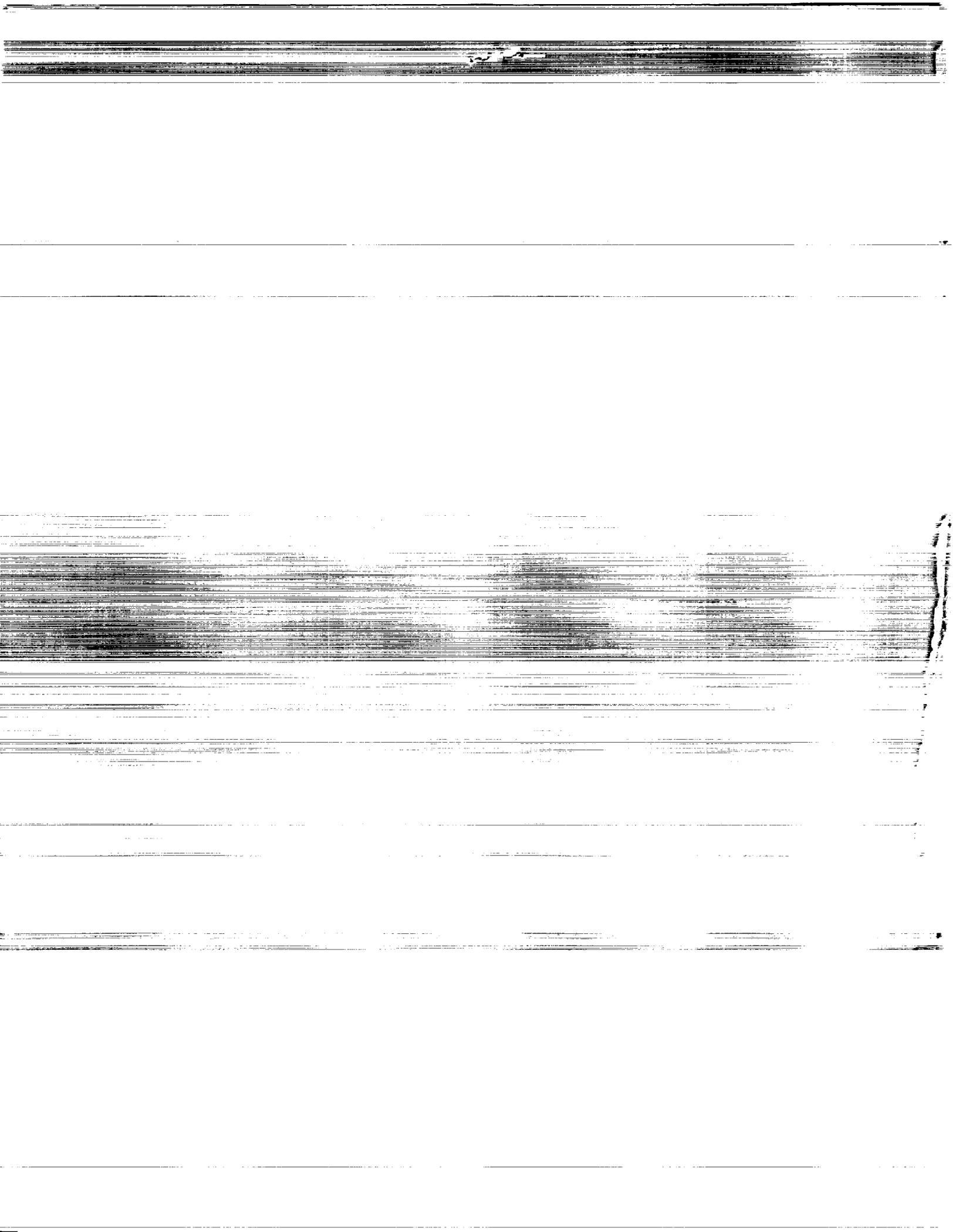
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# ANALYTIC THEORY FOR THE SELECTION OF SAFFMAN-TAYLOR FINGERS IN THE PRESENCE OF THIN FILM EFFECTS

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## Abstract

An analytic theory is presented for the width selection of Saffman-Taylor fingers in the presence of thin film effect. In the limit of small capillary number  $Ca$  and small gap to width ratio  $\epsilon$ , such that  $\epsilon \ll Ca \ll 1$ , it is found that fingers with relative width  $\lambda < \frac{1}{2}$  are possible such that  $\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} = k \frac{\epsilon}{Ca^{3/2}}$ , where the positive constant  $k$  depends on the branch of solution and equals 2.776 for the first branch. A fully nonlinear analysis is necessary in this problem even to obtain the correct scaling law. It is also shown how in principle, the selection rule for arbitrary  $Ca$  can be obtained.

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## I. Introduction

In recent years, considerable theoretical work, both numerical (see McLean & Saffman (1981), Romero (1982), Vanden-Broeck (1983), Kessler & Levine (1985), Tanveer (1986) and Tanveer (1987a)), and analytical (see, for example, Combescot et al (1986), Shraiman (1986), Hong & Langer (1986), Tanveer (1987b), Combescot et al (1988), Dorsey & Martin (1987), Combescot & Dombre (1988), Hong & Langer (1988) and Tanveer (1989a)) has been focussed on the singular effect of surface tension that breaks the continuum of Saffman-Taylor (Saffman & Taylor (1958), Taylor & Saffman (1959)) exact solutions for steadily propagating fingers and bubbles through a viscous fluid in a Hele-Shaw cell under the assumption that the thin film of the more viscous fluid in the narrow gap direction plays no significant role and that there is no variation of curvature in the transverse (narrow gap) direction. The above simplifying assumptions leads to a set of relatively simple boundary conditions on the flow variables at the interface and will be referred to as the McLean-Saffman (MS) boundary conditions henceforth. In this simplification, the mathematical equations for a finger contain a single non-dimensional parameter  $\mathcal{G} = \frac{b^2 T \pi^2}{48 \mu U a^2}$ , where  $b$  is the gap width and  $2a$  the cell width of the Hele-Shaw cell;  $T$ ,  $\mu$  and  $U$  are the surface tension, viscosity of the displaced fluid and the finger velocity respectively. The analytical and numerical evidence to date suggests that with the MS boundary conditions, for  $\mathcal{G}$  over some range, a discrete set of steady solutions are possible, each characterized by a different velocity. In the limit of zero  $\mathcal{G}$ , the finger or bubble velocity  $U \rightarrow 2V$  for all branches of solution, where  $V$  is the velocity of displaced viscous fluid at infinity. For the finger, with the MS boundary conditions, the continuity of fluid flow implies that its relative width  $\lambda = V/U$ , from which follows that as  $\mathcal{G} \rightarrow 0$ ,  $\lambda \rightarrow \frac{1}{2}$ . It is also known that the selection of the finger (Combescot et al (1986), Shraiman (1986) and Hong & Langer (1986)) or bubble velocity (see Combescot & Dombre (1988), Hong & Family (1988) and Tanveer (1989a)) as well as its symmetry (see Tanveer (1987b) and Combescot & Dombre (1988)) about the channel centerline for small  $\mathcal{G}$  is caused by transcendently

small corrections in  $\mathcal{G}$  that are beyond all orders of an asymptotic series in powers of  $\mathcal{G}$ .

It has also been shown that such terms play a crucial role in the linear stability (see Tanveer (1987c), Bensimon et al (1987), Tanveer (1989a)) as well (for the MS boundary conditions). When  $\mathcal{G} = 0$ , there is a continuum (see Tanveer (1987c), Tanveer & Saffman (1987)) of unstable modes for the finger or the bubble suggesting that the time dependent problem with zero surface tension is ill-posed. However, if  $\mathcal{G}$  is small but non zero, there are in general no eigenvalues close to the eigenvalues of the zero  $\mathcal{G}$  linear stability operator. In the small  $\mathcal{G}$  limit, it has been shown that this is due to the effect of transcendently small terms in  $\mathcal{G}$ . Numerical (see Kessler & Levine (1986), Bensimon (1986) and Tanveer & Saffman (1987)) and analytical (see Tanveer (1987c) and Tanveer (1989a)) calculations show that only one branch of finger solutions is stable, while others are unstable for arbitrary  $\mathcal{G}$ . However, numerical simulation by Degregoria & Schwartz(1986) of the time dependent problem based on the MS boundary conditions show instability at sufficiently small values of the surface tension parameter, despite the linear stability of these solutions. It is now understood (Bensimon (1986) that the instability observed in the time dependent calculations is a manifestation of all the branches of solution tending to the same  $\lambda = \frac{1}{2}$  Saffman-Taylor solution. Since all but one is unstable, the threshold amplitude of instability must tend to zero as surface tension tends to zero. Bensimon (1986) further conjectures that the threshold amplitude of instability tends to zero exponentially with surface tension and his numerical results tends to support this though accurate verification of the claim does not appear possible by direct numerical calculations. It has been suggested (Kessler & Levine, 1988) that the different branches of solution differ by an exponentially small amount as  $\mathcal{G}$  tends to zero and this would therefore explain the exponential dependence of threshold amplitude of instability. However, we disagree with this explanation since  $\lambda$  and the shapes on different solution branches differs by terms of order  $\mathcal{G}^{2/3}$  for small  $\mathcal{G}$ , which is not consistent with exponential dependence of the threshold level of instability, if indeed it is true as Bensimon suggests.

Aside from the point about the threshold level of instability raised in the last paragraph, the behavior of solutions to mathematical equations based on the McLean-Saffman boundary conditions appear to be quite well understood. However, experiments by Saffman & Taylor (1958), Tabeling et al (1987) and Kopf-Sill & Homsy (1987) with the finger show quantitative discrepancies with the theoretical finger width predictions of the MS theory. The bubble experiments (Maxworthy (1986), Kopf-Sill & Homsy (1988)) also show significant disagreements. Detailed experiments with the finger by Tabeling et al (1987) suggest that there isn't a single control parameter  $\mathcal{G}$  as would be the case if the MS theory were accurate. Experiments of Tabeling et al (1987) also show that to a significant extent, the gap to width ratio  $\epsilon = \frac{b}{2a}$  and capillary number  $Ca = \frac{\mu U}{T}$  are two independent control parameters. Equivalently, one can think of  $\mathcal{G}$  ( $= \frac{\epsilon^2 \kappa^2}{12Ca}$ ) and  $Ca$  as two independent parameters. This suggests that the thin film effects play a significant role in the fingering problem that have to be accounted for to explain discrepancies between the MS theory and experiment for the the finger width prediction. There is also a significant discrepancy on the onset of instability for small  $\mathcal{G}$ . Experiment (Tabeling et al, 1987) shows that the finger is unstable for small  $\mathcal{G}$ . However, there does not appear to be a sharp critical value of  $\mathcal{G}$  below which the finger is unstable; the instability point appearing to depend on the noise level in the experiment. This suggests perhaps a nonlinear instability mechanism (at least under some experimental conditions) as suggested by Bensimon (1986) for the MS boundary conditions. However, the threshold noise level that destabilizes the finger depends both on  $\mathcal{G}$  and  $Ca$  suggesting that the Bensimon mechanism for non-linear stability needs significant modifications if it is to be valid for boundary conditions that include thin film effects.

Saffman (1982) discusses the general form of the boundary conditions necessary to include the thin film effects. Park & Homsy (1985) and Reinelt (1987a) further develop details of these conditions that incorporate thin film effects into gap averaged 2-D boundary conditions for the steady finger, which we call Saffman-Park-Homsy-Reinelt (SPHR)

conditions. In addition to the pressure drop across the interface due to the lateral curvature of the interface that is included in the MS boundary conditions, the SPHR conditions include an additional pressure drop term due to variation of transverse curvature at different points on the interface in the lateral plane. It also includes terms in the kinematic boundary conditions that account for flow into the thin film region. The SPHR conditions, as we shall see in this paper, are quite complicated since they involve knowledge of functions that have to be determined by solving an associated 3-D problem in the transverse plane. In general, the solution to this associated three dimensional problem has to be computed numerically. For small  $Ca$ , using Bretherton (1961) results, Park & Homsy (1985) and Reinelt (1987a) have found asymptotic expressions for these functions. Numerical calculations by Reinelt (1987b) and Sarker & Jasnow (1987) show that the discrepancy between theory and experiment on the finger width dependence on the control parameters is greatly reduced when the SPHR boundary conditions are used. Equivalent boundary conditions for the bubble or for the time dependent problem are yet to be deduced.

The numerical work incorporating thin film effects cited in the last paragraph is important; however it leaves some other important questions unresolved. First is the existence of other branches of solutions. Reinelt (private communication) has found some other branches over some parameter ranges. For reasons stated earlier, it is important to know if all these steady state solutions tend to the same width as  $\mathcal{G} \rightarrow 0$ , or if it is possible for the limiting finger width to be different for different branches. Also, the precise dependence of  $\lambda$  on  $\mathcal{G}$  and  $Ca$  in the limit of small  $\mathcal{G}$  is difficult to conclude from the numerical solutions for we now have two control parameters. To compound our problem, the problem is numerically ill posed as  $\mathcal{G} \rightarrow 0$  for any  $Ca$  as a continuum of solution with arbitrary width exists for  $\mathcal{G} = 0$  and, as seen in this paper, transcendently small terms in  $\mathcal{G}$  have to be resolved for determination of  $\lambda$ . The need of an analytic theory to resolve this limit is therefore obvious.

The purpose of this paper is to incorporate the thin film effect into an analytic theory

which is valid for small  $\mathcal{G}$ . Further, we restrict ourselves primarily to the case of small  $Ca$ , where simplifications of the SPHR boundary conditions are possible by using Bretherton's formulae, though in section 7, we show how the analytic theory can be extended for any  $Ca$  by some subsidiary numerical calculations. Also, our primary concern will be the possibility of solution for  $\lambda < \frac{1}{2}$ , though we show that selection of finger width with  $\lambda > \frac{1}{2}$  is possible for a few restricted cases in the two-parameter space. We also present conditions under which the selection results of the theory based on MS boundary conditions hold. However, we do not present an exhaustive study of selection for all possible relative orderings of the small parameters  $Ca$  and  $\mathcal{G}$ .

For  $\lambda < \frac{1}{2}$ , we conclude that it is possible for solutions to exist for  $\mathcal{G} \ll Ca \ll 1$  with  $\frac{\lambda^2(1-\lambda)}{1-2\lambda} = \bar{k} \frac{\mathcal{G}^{1/2}}{Ca}$ , where the positive constant  $\bar{k}$  depends on the branch of solution and equals 3.061 for first 1st branch. In terms of control parameters  $\epsilon$  and  $Ca$ , this implies that for  $\epsilon \ll Ca \ll 1$ ,  $\frac{\lambda^2(1-\lambda)}{1-2\lambda} = k \frac{\epsilon}{Ca^{3/2}}$ , where the positive constant  $k$  depends on the branch of solution and equals 2.776 for the first branch. The lower limit on  $Ca$  for the validity of the above relation is sufficient to ensure that the solutions found satisfy the assumptions made in the analysis. Whether the relation hold over a larger range of the parameter space is an open question. We notice that when  $\mathcal{G}^{1/2} \ll Ca \ll 1$ ,  $\lambda$  is close to zero. On the otherhand if  $1 \gg \mathcal{G}^{1/2} \gg Ca \gg \mathcal{G}$ ,  $\lambda$  is close to a half. If  $\mathcal{G}^{1/2} = O(Ca)$ , for sufficiently small values of the two parameters  $Ca$  and  $\mathcal{G}$ , one can obtain fingers with width in the open interval  $(0, \frac{1}{2})$ . In this range of  $Ca$  and  $\mathcal{G}$ , the  $\lambda$  on different branches do not tend to the same value as  $Ca, \mathcal{G} \rightarrow 0$ .

For  $\lambda > \frac{1}{2}$ , we conclude that the finger solutions calculated on the basis of the MS theory for which  $\lambda \sim \frac{1}{2} + \text{constant } \mathcal{G}^{2/3}$  persists when  $Ca \ll \mathcal{G}^{7/3} \ll 1$ . This is rather unexpected since on inspection of the SPHR boundary conditions and their limiting form using Bretherton's formulae, it would appear that thin film or transverse curvature term or both (call them the 3-D effects) are significantly bigger than the lateral curvature term that is included in the MS theory when  $Ca \gg \mathcal{G}^3$ . Thus for  $\mathcal{G}^3 \ll Ca \ll \mathcal{G}^{7/3}$ ,

we have a situation where to the leading order, the solutions based on the MS theory gives the correct finger shape and the width scaling with  $\mathcal{G}$  even when the 3-D terms in the SPHR boundary conditions far exceeds the lateral curvature term on the finger boundary. The reason for this unexpected validity of the MS theory is, first, that both  $Ca$  and  $\mathcal{G}$  are small so that the deviation from the Saffman-Taylor finger solutions is actually small. Indeed, the role of terms such as lateral curvature and transverse curvature is not so much to change the Saffman-Taylor shapes as to determine the finger width which is arbitrary to zeroth order. Second, the finger width is being determined by transcendentally small terms in  $\mathcal{G}$  in the physical domain which can only be determined by analytic continuation of the equations to the neighborhood of some point in the unphysical plane that is the source of the transcendentally small correction. The relative size of lateral, transverse curvature and thin film leakage terms in an 'inner' region near this point is rather different from what they are in the physical domain; yet it is this relative size which determines the finger width. It turns out that the lateral curvature is far bigger in the 'inner' region than the thin film leakage or the transverse curvature term when  $Ca \ll \mathcal{G}^{7/3} \ll 1$  and that explains the unexpected persistence of the solutions based on the MS theory. However, in this paper, for this range of parameters, we do not address the question of existence of other kinds of solutions not found in the MS theory.

Our analysis also shows the importance of doing a nonlinear analysis in the region of nonuniformity of the original perturbation expansion. We point out that in the case of the McLean-Saffman boundary condition, nonlinear analysis is only necessary on an equation containing one parameter  $\frac{(\lambda-1/2)}{\mathcal{G}^{2/3}}$ . If the nonlinear equation is replaced by a linear equation, the linear equation still contains this parameter. Only the numerical value of  $\frac{(\lambda-1/2)}{\mathcal{G}^{2/3}}$  obtained by matching to the outer solution will be affected by the ad hoc linearization. Thus, only the proportionality constant between  $\lambda - \frac{1}{2}$  and  $\mathcal{G}^{2/3}$  is affected if one resorts to a simpler linearized analysis. With the thinfilm effects, there are, in general, nested inner regions surrounding the point of non-uniformity of the outer

perturbation expansion and the form of the nonlinear equations together with the type of parameters in each such region is determined by the nonlinearity of the equation in the next outer region. If the equations were linearized at the outset, one is unable to identify the correct scaling law between the three parameters  $\lambda$ ,  $\mathcal{G}$  and  $Ca$ , let alone determine the scaling constants.

## 2. Mathematical Formulation and leading order solution:

We consider the two-dimensional averaged flow in a Hele-Shaw cell of width  $2a$  where a finger of zero viscosity, moving with velocity  $U$ , displaces a viscous fluid of viscosity  $\mu$ . The 2-D averaged velocity field in the region not occupied by the finger is the gradient of a harmonic potential function  $\phi$ . The finger boundary is assumed to be asymptotically parallel to the walls. Here, in this paper, we will take each of  $a$  and  $U$  to be unity without any loss of generality since this is equivalent to non-dimensionalizing all our variables using these two parameters. We will now proceed with the understanding that all the variables have been non-dimensionalized. The flow domain in the frame of the steady finger, the  $z = x + iy$  plane, is shown in Fig. 1. We introduce the complex velocity potential,  $W = \phi + i\psi$ , which will be an analytic function of the complex variable  $z$  in our flow domain. In the frame of the steadily moving finger, the boundary conditions on the finger that incorporate the the transverse thin film effect (see Reinelt (1987b) for details) are:

$$\phi + x = -\frac{4\mathcal{G}}{\pi^2} \left[ \sqrt{\frac{\pi^2}{12}} Ca^{-1/2} \mathcal{G}^{-1/2} \kappa^0(Ca n_x) + \kappa^1(Ca n_x) \frac{1}{R} \right] \quad (2.1)$$

$$\psi_\nu = - \left[ m^0(Ca n_x) + m^1(Ca n_x) \sqrt{\frac{12}{\pi^2}} \mathcal{G}^{1/2} Ca^{1/2} \frac{1}{R} \right] y_\nu \quad (2.2)$$

where  $\mathcal{G} = \frac{b^2 \pi^2}{48\mu U a^2}$ ,  $Ca = \frac{\mu U}{\sigma}$  (capillary number),  $n_x = \cos \theta$  is the component of the outward unit normal to the finger (see Fig. 1) along the  $x$ -axis,  $\nu$  is some parameter in the interval  $(0, \pi)$  (specific choice given later) parametrizing the finger boundary and the subscript with respect to  $\nu$  denotes derivative with respect to  $\nu$ .  $R$  is the radius of curvature of the interface in the lateral ( $x-y$ ) plane. The functions  $\kappa^0$ ,  $\kappa^1$ ,  $m^0$  and

$m^1$  have been calculated by Reinelt (1987a) numerically over a range of arguments. The appearance of  $n_x$  in the argument of these functions is because only the normal component of the finger velocity is relevant in the thin film effect. In the limit of small  $Ca$ , following Bretherton, Reinelt found that

$$\kappa^0(Ca) = \text{constant} - 3.878 Ca^{2/3} + \dots \quad (2.3)$$

$$\kappa^1(Ca) = -\pi/4 + 4.153 Ca^{2/3} + \dots \quad (2.4)$$

$$m^0(Ca) = 1.3375 Ca^{2/3} + \dots \quad (2.5)$$

$$m^1(Ca) = -1.3375 \frac{\pi}{4} Ca^{2/3} \quad (2.6)$$

The values of the constant coefficients in (2.3) and (2.4) are different in their third significant figures from those originally quoted in Reinelt (1987b). Reinelt (private communication) found these revised values after a more accurate calculation than previously reported. Note that the constant term in (2.3) does not affect the finger shape or the fluid velocity field since it can be always absorbed as part of  $x$  on the left hand side of (2.1) by suitably choosing the origin of  $x$ . Thus from this point onwards, we will neglect the constant term on the right hand side of (2.3). The MS theory corresponds to neglecting all other terms involving powers of  $Ca$ , while replacing  $\kappa^1$  by  $-1$ . A rational approximation when  $Ca \ll 1$  and  $Ca \ll \mathcal{G}^3$  would be to replace  $\kappa^1$  by  $-\frac{\pi}{4}$  while neglecting all terms in (2.1) and (2.2). However, this only modifies the sole control parameter  $\mathcal{G}$  of the MS theory by a factor of  $\frac{\pi}{4}$ . Also the range of  $Ca$  and  $\mathcal{G}$  investigated in the experiments to date, it is clear that  $Ca$  is not small enough to justify this approximation.

The mathematical formulation here is a generalization of the previous formulation (Tanveer, 1987b) developed for the MS boundary conditions. In the previous work with the MS boundary conditions, no assumption was made on the symmetry of the finger about the channel centerline. As part of the conclusion, it was found that only symmetric fingers are possible for non zero  $\mathcal{G}$ . In this case, for the sake of simplicity, we will assume a priori

that the finger is symmetric about the channel centerline. The existence of nonsymmetric finger with the SPHR boundary conditions is still an open question.

Consider the conformal map of the flow domain in the  $z$ -plane into the interior of the unit semi-circle in the  $\zeta$ -plane (Fig. 2) such that  $z = \infty$ ,  $z = -\infty + i$  and  $z = -\infty - i$  are mapped to  $\zeta = 0$ ,  $+1$  and  $-1$ , respectively. The finger boundary is mapped to the arc of the semi-circle in the  $\zeta$ -plane. Introduce analytic functions  $f(\zeta)$  and  $g(\zeta)$  in the unit  $\zeta$ -semicircle so that

$$z = -\frac{2}{\pi} \ln \zeta + \frac{2}{\pi} (1 - \lambda) \ln (\zeta^2 - 1) - i (1 - 2\lambda) + \frac{2}{\pi} f(\zeta) \quad (2.7)$$

$$W(\zeta) = -2 \frac{(1 - \lambda)}{\pi} \ln (\zeta^2 - 1) + \frac{2(1 - \lambda + \alpha)}{\pi} \ln \zeta + i (1 - \lambda - \alpha) + \frac{2}{\pi} g(\zeta) \quad (2.8)$$

where

$$\alpha = - \int_0^{\pi/2} d\nu \left[ m^0 + m^1 \sqrt{\frac{12}{\pi^2}} \mathcal{G}^{1/2} C a^{1/2} \frac{1}{R} \right] y_\nu d\nu \quad (2.9)$$

where  $\zeta = e^{i\nu}$  on the semi-circular boundary. Note  $\alpha$  accounts for the total flow across the interface edge. From conservation of fluid flow, it is easy to show that the fluid velocity  $V$  far ahead of the finger in the laboratory frame is given by  $V = (\lambda - \alpha) U$ . Then the condition that the real diameter of the unit semi-circle in the  $\zeta$  plane corresponds to the walls of the cell where  $Im z = \pm 1$  implies

$$Im f = 0 \quad (2.10)$$

The condition that on the upper and lower cell walls  $Im W = \mp(1 - \lambda + \alpha)$  implies

$$Im g = 0 \quad (2.11)$$

The boundary conditions (2.1) and (2.2) can then be written as the following boundary conditions on  $f$  and  $g$  on the semi-circular boundary  $\zeta = e^{i\nu}$ , where  $\nu$  is in the interval  $(0, \pi)$ :

$$Re (f + g) = - 3^{-1/2} \mathcal{G}^{1/2} C a^{-1/2} \kappa^0 - \frac{2\mathcal{G}}{\pi} \frac{\kappa^1}{R} \quad (2.12)$$

$$\operatorname{Re}(\zeta g') = -m^0 \operatorname{Re} \zeta (f' + h) - m^1 \sqrt{\frac{12}{\pi^2}} \mathcal{G}^{1/2} C a^{1/2} \frac{1}{R} \operatorname{Re} \zeta (f' + h) - \alpha \quad (2.13)$$

where primes denote derivatives with respect to the argument and

$$h(\zeta) = \frac{(1 - p^2 \zeta^2)}{\zeta(\zeta^2 - 1)} \quad (2.14)$$

where

$$p^2 = 2\lambda - 1$$

is in the open interval  $(-1, 1)$  for  $\lambda$  in the open interval  $(0, 1)$ . Note that  $\alpha$  can be expressed as:

$$\alpha = -\frac{2}{\pi} \int_0^{\pi/2} d\nu \left[ m^0 + m^1 \sqrt{\frac{12}{\pi^2}} \mathcal{G}^{1/2} C a^{1/2} \frac{1}{R} \right] \operatorname{Re} e^{i\nu} [f'(e^{i\nu}) + h(e^{i\nu})] \quad (2.15)$$

Also note that owing to Schwartz's reflection principle each of (2.12) and (2.13) hold in the lower half semi-circular boundary as well.

One can establish that on  $\zeta = e^{i\nu}$ ,  $\nu$  in the interval  $(0, \pi)$ ,

$$n_{\pm} = -\frac{1}{|f' + h|} \operatorname{Re} [\zeta (f' + h)] \quad (2.16)$$

$$\frac{1}{R} = -\frac{\pi}{2|(f' + h)|} \operatorname{Re} \left[ 1 + \zeta \frac{f'' + h'}{f' + h} \right] \quad (2.17)$$

There is an alternate way of writing each of (2.16) and (2.17) which defines analytic continuation of  $n_{\pm}(\zeta)$  and  $\frac{1}{R}(\zeta)$  off the arc of the upper half unit  $\zeta$  semi-circle:

$$n_{\pm}(\zeta) = \frac{i}{2} \frac{l_2(\zeta) - l_1(\zeta)}{(l_1(\zeta))^{1/2} (l_2(\zeta))^{1/2}} \quad (2.18)$$

$$\frac{1}{R}(\zeta) = \frac{\pi i (\zeta^2 - 1) \left[ -1 + \frac{1}{2} \frac{\zeta l_2'(\zeta)}{l_2(\zeta)} + \frac{1}{2} \frac{\zeta l_1'(\zeta)}{l_1(\zeta)} \right]}{2 (l_1(\zeta))^{1/2} (l_2(\zeta))^{1/2}} \quad (2.19)$$

where

$$l_1(\zeta) = \zeta^2 - p^2 + \frac{1}{\zeta} f'(1/\zeta) \quad (2.20)$$

$$l_2(\zeta) = 1 - p^2 \zeta^2 + \zeta(\zeta^2 - 1) f'(\zeta) \quad (2.21)$$

For  $Ca = O(1)$  and  $\mathcal{G} \ll 1$ , the leading order approximation can be found by setting  $\mathcal{G} = 0$  in (2.12) and (2.13). Thus, to the leading order,  $g = -f_0$ ,  $f = f_0$ , where  $f_0$  is determined by the boundary condition

$$Im f_0 = 0 \quad (2.22)$$

on the diameter of the unit semi-circle and on the arc of the unit circle,  $\zeta = e^{i\nu}$ ,

$$Re \zeta f'_0 = m^0 (Ca n_x^0(\zeta)) Re [\zeta (f'_0 + h)] + \alpha^0 \quad (2.23)$$

where  $n_x^0$  is  $n_x$  with the substitution  $f = f_0$  in (2.16) and  $\alpha^0$  is the constant  $\alpha$  as determined by (2.15) when  $m^1$  is neglected and  $Ca n_x^0$  is substituted as the argument of  $m^0$ . The solution to (2.23) was found numerically for arbitrary  $p^2$  in the open interval  $(-1, 1)$ , i.e.  $\lambda$  in  $(0, 1)$ , by expanding  $f_0$  in a power series

$$f_0 = a_1^0 \zeta^2 + a_2^0 \zeta^4 + .. \quad (2.24)$$

truncating the series to  $N$  terms and satisfying (2.23) at  $N$  uniformly spaced out points on the arc of the unit  $\zeta$  quarter circle. The condition (2.10) is automatically satisfied by restricting to real  $a_n^0$ . The resulting system of nonlinear algebraic equations was solved using Newton iteration and consistency of results checked by doubling  $N$ .

Higher order formal corrections to the perturbation expansion in powers of  $\mathcal{G}^{1/2}$  can be found by expanding  $f$  and  $g$  as:

$$f = f_0 + \mathcal{G}^{1/2} f_1 + .. \quad (2.25)$$

$$g = g_0 + \mathcal{G}^{1/2} g_1 + .. \quad (2.26)$$

where  $f_0$  and  $g_0$  are as determined above.  $f_1$  and  $g_1$  are determined by satisfying equations:

$$Im f_1 = 0 \quad (2.27)$$

$$Im g_1 = 0 \quad (2.28)$$

on the real diameter and on the arc of the unit semi-circle in the  $\zeta$  -plane,

$$Re (f_1 + g_1) = -3^{-1/2} Ca^{-1/2} \kappa^0(n_x^0 Ca) \quad (2.29)$$

$$\begin{aligned} Re \zeta g'_1 + m^0 (Ca n_x^0) Re \zeta f'_1 + m^{0'} (Ca n_x^0) Ca Re \zeta (f'_0 + h) n_x^1 \\ = -m^1 (Ca n_x^0) \sqrt{\frac{12}{\pi^2}} Ca^{1/2} \frac{1}{R_0} Re \zeta (f'_0 + h) - \alpha_1 \end{aligned} \quad (2.30)$$

where

$$n_x^1 = -\frac{1}{|f'_0 + h|} Re \zeta f'_1 + \frac{Re \zeta (f'_0 + h)}{|f'_0 + h|} Re \left( \frac{f'_1}{f'_0 + h} \right)$$

For each  $\lambda$  and  $Ca$ , by truncating power series representations in powers of  $\zeta^2$  for each of  $f_1$  and  $g_1$  to a finite number of terms and satisfying (2.29) and (2.30) at a discrete set of uniformly spaced out points on the unit  $\zeta$  quarter circle, we solved the resulting system of linear algebraic equations. The resulting  $a_n$  and  $b_n$  were found to decay with  $n$  for large  $n$  and there was consistency on doubling the collocation points and unknowns. This suggests that a regular perturbation expansion fails to select  $\lambda$  in contradiction to direct numerical results of Reinelt (1987b). A similar contradiction occurred earlier in the context of McLean-Saffman boundary conditions (McLean & Saffman, 1981) which was later resolved (Combescot et al (1986), Shraiman (1986) and Hong & Langer (1986)) by extracting transcendentally small terms in  $\mathcal{G}$ .

When  $Ca \ll 1$ , the absence of selection from terms of a regular perturbation expansion can be more clearly demonstrated. Since the right hand side of each of (2.12) and (2.13) are small for small  $Ca$  and  $\mathcal{G}$  as is clear from (2.3)-(2.6), to the leading order,  $f = 0$  and  $g = 0$ , i.e. we get the Saffman Taylor finger solutions in which  $\lambda$  and hence  $p^2$  is arbitrary. Now, if we were to try to improve this approximation by including the next order correction, we would substitute  $f = 0$  and  $g = 0$  into each term on the right hand side of (2.12) and (2.13) and determine the first non trivial  $g$  and  $f$  term on the left hand side. One can systematically proceed to determine each of  $f$  and  $g$  in a perturbation expansion involving powers of  $\mathcal{G}^{1/2}$  and  $Ca^{1/6}$ , the precise

form depending on the relative ordering of the two small parameters. This can clearly be done again without any constraint on  $p^2$ , since solving at each stage corresponds to calculating two harmonic functions  $Re(f_n + g_n)$  and  $Re g_n$  with specified Dirichlet and Neumann data respectively on the unit circular boundary where the Neumann data consistency condition is easily seen to be satisfied. The solution  $g_n$  is only determined to an arbitrary additive constant that does not affect fluid flow or the finger shape. There are no sources of nonuniformity of this outer perturbation expansion except possibly near  $\zeta = \pm 1$ , i.e. the tail of the finger. For the MS boundary conditions, McLean & Saffman (1981) have shown that a secondary expansion near the tail is possible and that this matches with the outer expansion without any constraint on  $\lambda$ . We expect that such a matching is possible in this problem as well. We will assume this is indeed the case. Thus, given this assumption, it is clear that  $\lambda$  remains undetermined by a regular perturbation series.

Thus, to calculate  $\lambda$  analytically for any  $Ca$ , one must calculate terms for  $f$  and  $g$  that are transcendentally small in  $\mathcal{G}$ . These are terms beyond all orders of the regular perturbation expansion in powers of  $\mathcal{G}$  and are subdominant except when every term of the regular perturbation expansion is zero. Kruskal & Segur (1986), in their pioneering work in calculating terms beyond all orders for a model third order nonlinear ODE, give a careful account of the issues involved in calculating terms beyond all orders.

As with the earlier case of MS boundary conditions, it will turn out that there is some open interval on the imaginary  $\zeta$  axis containing  $i$  where the condition of a finger boundary with a smooth tip,  $Im f = 0$ , is satisfied by every term of the regular perturbation series of  $f$  in powers of  $\mathcal{G}^{1/2}$  for any  $\lambda$ ; yet the leading order transcendentally small term in  $\mathcal{G}$  violates this condition except when  $\lambda$  is restricted to a discrete set of values.

We note that to find transcendentally small terms in the physical domain, we must find the source of non-uniformity of the regular perturbation expansion in the unphysi-

cal domain. Such sources of non-uniformity of the perturbation expansion contribute to transcendently small terms in the physical domain. One needs to rescale dependent and independent variables in the immediate neighborhood of the source of non-uniformity and construct an inner perturbation expansion. An inner-outer matching is then carried out and the terms that are beyond all orders in the matching procedure are transcendently small in the outer region that includes the physical domain. In this paper, only the leading order inner and outer matching is carried out since only the leading order transcendently small terms will be found. As a first step, it is necessary to locate the source of nonuniformity of the regular perturbation expansion and construct an inner expansion near the source of non-uniformity. For that purpose, it is necessary to continue analytically each of (2.12) and (2.13) outside the unit  $\zeta$  circle across the arc of the upper half semi-circle.

### 3. Analytical continuation of the equations to $|\zeta| > 1$

First, we note that from Poisson's integral formula relating a harmonic function and its conjugate inside the unit circle to its value on the boundary, one finds that for any analytic function  $F(\zeta)$  in the unit semi-circle with vanishing imaginary part on the real diameter,

$$F(\zeta) = \frac{1}{2\pi i} \int_C \frac{d\zeta'}{\zeta'} \left\{ \frac{\zeta + \zeta'}{\zeta' - \zeta} + \frac{1 + \zeta\zeta'}{1 - \zeta\zeta'} \right\} \text{Re } F(\zeta') \quad (3.1)$$

where the contour of integration  $C$  in the  $\zeta'$  plane is along the semi-circular arc from  $+1$  to  $-1$ . For convenience, we define the operation on  $\text{Re } F$  on the right hand side of (3.1) as  $\mathcal{I}(\text{Re } F)$ . If  $\text{Re } F(\zeta') = G(\zeta')$  for  $\zeta'$  on the unit semi-circular arc, where  $G(\zeta)$  is an analytic function defined off the semi-circular arc as well, then by deforming the contour in the  $\zeta'$  plane, one finds that the analytic continuation of  $F(\zeta)$  across the semi-circular arc for  $|\zeta| > 1$  must be

$$F(\zeta) = \mathcal{I}(G) + 2 G(\zeta) \quad (3.2)$$

Note that on the arc of the unit circle,  $\zeta^{**} = \zeta^{-1}$  and  $\text{Re } F = \frac{1}{2}[F(\zeta) + F(\zeta^{-1})]$  for any function  $F$  which has vanishing imaginary part on the real diameter of the unit

semi-circle. Using this and the continuation property (3.2), one finds that the analytic continuation of (2.12) and (2.13) to  $|\zeta| > 1$  across the arc of the upper half semi-circle must be

$$f + g = h_1(\zeta) - \sqrt{\frac{4\mathcal{G}}{3Ca}} \kappa^0(Ca n_x(\zeta)) + h_2(\zeta) - \frac{4\mathcal{G}}{\pi} \frac{1}{R}(\zeta) \kappa^1(Ca n_x(\zeta)) \quad (3.3)$$

$$\begin{aligned} \zeta g' = & h_3(\zeta) - \frac{l_2(\zeta) - l_1(\zeta)}{\zeta^2 - 1} m^0(Ca n_x(\zeta)) + h_4(\zeta) \\ & - \sqrt{\frac{12\mathcal{G}Ca}{\pi^2}} \frac{1}{R}(\zeta) \frac{l_2(\zeta) - l_1(\zeta)}{\zeta^2 - 1} m^1(Ca n_x(\zeta)) - \alpha \end{aligned} \quad (3.4)$$

where we assume each of the functions  $m^0$ ,  $m^1$ ,  $\kappa^0$  and  $\kappa^1$  are locally analytic functions of their argument and

$$h_1(\zeta) = -\sqrt{\frac{\mathcal{G}}{3Ca}} \mathcal{I}(\kappa^0) \quad (3.5)$$

$$h_2(\zeta) = -\frac{2\mathcal{G}}{\pi} \mathcal{I}\left(\kappa^1 \frac{1}{R}\right) \quad (3.6)$$

$$h_3(\zeta) = -\frac{1}{2} \mathcal{I}\left(m^0 \frac{l_2 - l_1}{\zeta^2 - 1}\right) \quad (3.7)$$

$$h_4(\zeta) = -\sqrt{\frac{3\mathcal{G}Ca}{\pi^2}} \mathcal{I}\left(m^1 \frac{1}{R} \frac{l_2 - l_1}{\zeta^2 - 1}\right) \quad (3.8)$$

Note that each of the functions  $h_1$ ,  $h_2$ ,  $h_3$  and  $h_4$  involve integration of values of functions  $f$  and  $g$  on the arc of the unit  $\zeta'$  circle and therefore analytic everywhere in  $|\zeta| > 1$ .

Equations (3.3) and (3.4) are a pair of nonlinear integro differential equations for  $f$  and  $g$  that appears too formidable to be of any practical utility. However, it is amenable to simplifications when  $\mathcal{G}$  is small. When  $|\zeta| > 1$ ,  $|\zeta^{-1}| < 1$  and so terms such as  $f'(1/\zeta)$ ,  $f''(1/\zeta)$  appearing in  $\frac{1}{R}$  and  $n_x$  in (2.18) and (2.19) can be replaced by the regular perturbation (2.25) even when the deviations of  $f(\zeta)$  from the regular perturbation series (2.25) is not small as is true near the point of nonuniformity of the regular perturbation series. To the leading order, it is appropriate to replace  $f'(1/\zeta)$  by  $f'_0(1/\zeta)$  and  $f''(1/\zeta)$  by  $f''_0(1/\zeta)$ . The same substitution is appropriate for  $f$  appearing

in the integrand in each of  $h_1$  through  $h_4$ . Then the pair of equations (3.3) and (3.4) become a pair of nonlinear second order differential equation for  $f$  and  $g$ . We are interested in solutions to these equations that match with (2.25) and (2.26) as one moves away from the points of nonuniformity of the regular perturbation expansion towards the arc of the upper half unit semi-circle.

#### 4. Nature of transcendental correction

We now seek the form of the leading order transcendently small correction terms in  $\mathcal{G}$  to the regular expansion (2.25) and (2.26). As has been discussed before (see, for example, Kruskal & Segur (1986), Combescot et al (1986), Tanveer (1987b), Combescot et al (1988), Combescot & Dombre (1988) and Tanveer (1989)) linearizing the nonlinear equations about the leading order behavior given in (2.25) and (2.26) and looking for WKB type of solutions to the homogeneous part of the linearized equation.

In our case, the resulting homogeneous equations arising by linearizing (3.3) and (3.4) about  $f = f_0 + \mathcal{G}^{1/2} f_1$  are:

$$f_H + g_H = -\frac{Q_1}{L} \mathcal{G}^{1/2} f'_H - \frac{\mathcal{G}}{L} [f''_H + Q_2 f'_H] \quad (4.1)$$

$$\zeta g'_H = \zeta Q_3 f'_H + \zeta \frac{Q_4}{L} \mathcal{G}^{1/2} [f''_H + Q_2 f'_H] \quad (4.2)$$

where

$$Q_1 = \frac{l_{2_0} + l_{1_0}}{4\kappa^1 \zeta (\zeta^2 - 1)} \left( \sqrt{\frac{4Ca}{3}} \kappa^{0'} (Ca n_x^0) + \frac{4}{\pi} \frac{\mathcal{G}^{1/2}}{R_0} Ca \kappa^{1'} (Ca n_x^0) \right) \quad (4.3)$$

$$L = \frac{-i l_{1_0}^{1/2} l_{2_0}^{3/2}}{\kappa^1 (Ca n_x^0(\zeta)) \zeta^2 (\zeta^2 - 1)^2} \quad (4.4)$$

$$Q_2 = \frac{3\zeta^2 - 1}{\zeta (\zeta^2 - 1)} - \frac{3}{2} \frac{l'_{2_0}}{l_{2_0}} + \frac{1}{\zeta} - \frac{1}{2} \frac{l'_{2_0} (1/\zeta)}{l_{1_0}} \quad (4.5)$$

$$Q_3 = -m^0(Ca n_x^0(\zeta)) - \sqrt{\frac{12 \mathcal{G} Ca}{\pi^2}} \frac{1}{R_0} m^1(Ca n_x^0(\zeta)) - Ca \frac{i}{4} \frac{(l_{2_0} + l_{1_0})}{l_{2_0}^{3/2} l_{1_0}^{1/2}} (l_{2_0} - l_{1_0}) \left\{ m^{0'}(Ca n_x^0) + \sqrt{\frac{12 \mathcal{G} Ca}{\pi^2}} \frac{1}{R_0} m^{1'}(Ca n_x^0(\zeta)) \right\} \quad (4.6)$$

$$Q_4 = -\sqrt{\frac{12 Ca}{16}} \frac{l_{2_0} - l_{1_0}}{\zeta(\zeta^2 - 1)} \frac{m^1(Ca n_x^0(\zeta))}{\kappa^1(Ca n_x^0(\zeta))} \quad (4.7)$$

$$\frac{1}{R_0} = \frac{\pi i(\zeta^2 - 1) \left[ -1 + \frac{1}{2} \frac{\zeta'_{2_0}(\zeta)}{l_{2_0}(\zeta)} + \frac{1}{2} \frac{\zeta'_{1_0}(\zeta)}{l_{1_0}(\zeta)} \right]}{2 (l_{1_0}(\zeta))^{1/2} (l_{2_0}(\zeta))^{1/2}} \quad (4.8)$$

where the zero subscript on each term here refers to the evaluation of that term with the the substitution  $f = f_0 + \mathcal{G}^{1/2} f_1$ . Eliminating  $g_H$  between (4.1) and (4.2), one obtains the following second order linear homogeneous equation for  $f'_H$

$$f'''_H + \left( Q_2 + \frac{Q_4}{\mathcal{G}^{1/2}} + \frac{Q_1}{\mathcal{G}^{1/2}} - \frac{L'}{L} \right) f''_H + f'_H \left( Q'_2 - \frac{L'}{L} Q_2 + \frac{Q'_1}{\mathcal{G}^{1/2}} - \frac{Q_1 L'}{L \mathcal{G}^{1/2}} + \frac{L}{\mathcal{G}} (1 + Q_3) + \frac{Q_4}{\mathcal{G}^{1/2}} Q_2 \right) = 0 \quad (4.9)$$

In the limit of  $\mathcal{G} \rightarrow 0$ , one can seek WKB solution of the type

$$f'_H = e^{\mathcal{G}^{-1/2} w_0 + w_1 + o(\mathcal{G}^{1/2})} \quad (4.10)$$

to find that

$$W'_0 = -\frac{Q_4 + Q_1}{2} \pm \sqrt{\frac{(Q_4 + Q_1)^2}{4} - L(1 + Q_3)} \quad (4.11)$$

$$W'_1 = \frac{-W''_0 - W'_0(Q_2 - \frac{L'}{L}) - Q'_1 + Q_1 \frac{L'}{L} - Q_4 Q_2}{2 W'_0 + Q_4 + Q_1} \quad (4.12)$$

Let us define  $\tilde{g}_1$  and  $\tilde{g}_2$  as the asymptotic solution in (4.10) corresponding to the two different choices of the sign of the squareroot in (4.11). Thus

$$\tilde{g}'_1 = e^{\mathcal{G}^{-1/2} w_0^+ + w_1^+} \quad (4.13)$$

and

$$\tilde{g}'_2 = e^{\mathcal{G}^{-1/2} w_0^- + w_1^-} \quad (4.14)$$

where the plus and minus superscript on  $W_0$  and  $W_1$  correspond to the choice of plus or minus sign in (4.11). Further, to the order of approximation that the WKB solutions (4.13) and (4.14) are valid, it appropriate to neglect  $\mathcal{G}^{1/2}$  in the zero subscripted terms in  $Q_1, Q_2, Q_3, Q_4$  and  $L$  appearing in the right hand side of (4.12) but not (4.11).

The condition  $Re W_0^+ = 0$  and  $Re W_0^- = 0$  (called the Stokes lines) divide the upper half  $\zeta$  plane into different sectors. Including the leading order transcendental correction in the description of the asymptotic behavior of  $f$  in some sector of the complex  $\zeta$  plane adjoining the arc of the upper half unit semi-circle  $|\zeta| = 1$ ,  $Im \zeta > 0$ , we can write

$$f = f_0 + \mathcal{G}^{1/2} f_1 + \dots + C_1 \tilde{g}_1 + C_2 \tilde{g}_2 \quad (4.15)$$

where the transcendental term  $C_1 \tilde{g}_1 + C_2 \tilde{g}_2$  is assumed small. It is generally subdominant to every other term of the perturbation solution denoted symbolically by  $\dots$  in the above equation. If  $Re W_0^+ > 0$   $\tilde{g}_1$  is transcendentally large. If  $Re W_0^- > 0$ ,  $\tilde{g}_2$  is transcendentally large. In each case, the corresponding coefficient  $C_1$  or  $C_2$  has to be zero since each of  $Re W_0^+$  and  $Re W_0^-$  increase without bounds as  $\zeta = \pm 1$  or  $\zeta = 0$  is approached as can be seen from the integration of (4.11) once expressions for  $Q_1$ ,  $Q_4$  and  $L$  are substituted. As we cross any Stokes line, the coefficients  $C_1$  or  $C_2$  in (4.15) can change. The actual determination of the coefficients  $C_1$  and  $C_2$  is possible only by matching an inner perturbation expansion carried out in the neighborhood of points where the outer perturbation expansion (2.25) and (2.26) to (3.3) and (3.4) as well as the leading order WKB solutions (4.13) and (4.14) to (4.9) are invalid. This is rather an involved procedure for arbitrary  $Ca$  since the exact solution for  $f_0$  is not known and even the source of nonuniformity of the outer perturbation expansion has to be determined by a nontrivial numerical procedure. Further, no analytical formulae for the functions  $m^0$ ,  $m^1$ ,  $\kappa^0$  and  $\kappa^1$  exist for arbitrary values of their argument. In section 7, we will show more precisely what pieces of information would complete the selection theory for arbitrary  $Ca$  and give indications on how they might be obtained.

From now on till section 7, we will limit ourselves to small  $Ca$  and assume the formulae (2.3)-(2.6) to be valid. In this case, as noted earlier, the Saffman-Taylor exact finger solution  $f = 0$  and  $g = 0$  provides a convenient starting point. Higher order approximations involve powers of  $Ca^{1/6}$  and  $\mathcal{G}^{1/2}$ , the precise form depending on the

relative ordering of the two small parameters  $Ca$  and  $\mathcal{G}$ . We will symbolically denote the leading order correction with algebraic dependence on  $Ca$  and  $\mathcal{G}$  by  $\tilde{f}_1$ . Thus, away from the points of nonuniformity of the regular perturbation expansion, which in this case are at  $\zeta = \pm \frac{1}{p}$ ,

$$f \sim \tilde{f}_1 \quad (4.16)$$

To this, one can add terms with higher order algebraic dependence on  $Ca$  and  $\mathcal{G}$ . The behavior of the analytic continuation of  $\tilde{f}_1$  in the region  $|\zeta| > 1$  can be obtained from (3.3) and (3.4) by setting  $f'$  and  $f''$  to zero in every term involving these on the right hand side of (3.3) and (3.4) after the Bretherton approximations (2.3)-(2.6) are invoked. One can see that  $\tilde{f}_1$  is singular at  $\zeta = \pm 1/p$  which are then the sources of nonuniformity of the regular perturbation expansion. We also notice that  $\tilde{f}_1$  is real on some open interval on the imaginary axis that includes  $\zeta = i$ . The same can be shown to be true to every order of the regular perturbation expansion. However, transcendentally small corrections to (4.16), as we shall see in a moment, do not generally satisfy this condition implying that the finger tip is not smooth for arbitrary  $\lambda$ .

Terms that are transcendentally small in  $\mathcal{G}$ , which to the leading order are constant multiples of  $\tilde{g}_1$  or  $\tilde{g}_2$  as determined by (4.13) and (4.14), can be added to (4.16). When  $|1 - p^2 \zeta^2| \gg Ca^{2/7} (\Gamma^{-\lambda})$ , the expression for the asymptotic WKB solutions to (4.9) simplify even further because the capillary number  $Ca$  has been assumed to be small and so each of  $Q_1$ ,  $Q_2$ ,  $Q_3$  and  $Q_4$  can be neglected. In that region, to the same order that (4.13) and (4.14) are asymptotic solutions,  $\tilde{g}'_1$  and  $\tilde{g}'_2$  can be replaced by  $g'_1$  and  $g'_2$  where  $g_1$  and  $g_2$  are defined as

$$g_{1,2} = \frac{(1 - p^2 \zeta^2)^{3/8}}{(\zeta^2 - p^2)^{3/8}} e^{\pm \frac{i^{3/2}}{p^{1/2}} \int_{1/p}^{\zeta} d\zeta \frac{(1 - p^2 \zeta^2)^{3/4} (\zeta^2 - p^2)^{1/4}}{\zeta (\zeta^2 - 1)}} \quad (4.17)$$

Note that  $g_1$  and  $g_2$  are transcendentally small or large depending on the sign of  $Re P$ , where

$$P = i^{3/2} \int_{1/p}^{\zeta} d\zeta \frac{(1 - p^2 \zeta^2)^{3/4} (\zeta^2 - p^2)^{1/4}}{\zeta (\zeta^2 - 1)} \quad (4.18)$$

The Stokes lines here are the same as for the Mclean-Saffman boundary conditions as discussed earlier (Tanveer, 1987b). We have different cases depending on  $p^2 < 0$  or  $p^2 > 0$ . The Stokes lines in each case are shown in Figs. 3 and 4. Note that  $p^2 < 0$  corresponds to  $\lambda < \frac{1}{2}$  and  $p^2 > 0$  corresponds to  $\lambda > \frac{1}{2}$ .

### 5. Restriction to $\lambda < \frac{1}{2}$

It is appropriate in this case to define  $q$ , a positive real parameter, so that  $p = -iq$ . In that case the point of nonuniformity of the regular perturbation expansion in the upper half plane  $\zeta = \frac{1}{p} = \frac{i}{q}$  as shown in Fig. 3.

Away from the immediate neighborhood of  $\zeta = \frac{i}{q}$ , the desired asymptotic behavior for  $f$  must be

$$f = \tilde{f}_1 + \dots + C_1 g_1 \tag{5.1}$$

in sector I of Fig. 3; in sector II, one must have

$$f = \tilde{f}_1 + \dots + C_2 g_2 \tag{5.2}$$

and in sector III

$$f = \tilde{f}_1 + \dots + C_3 g_2 \tag{5.3}$$

because  $g_1$  is transcendentally small in sector I, and  $g_2$  transcendentally small in sectors II and III. No transcendentally large terms are allowed in this three sectors as each of the sectors I, II and III extend all the way to the physical region inside the unit upper half  $\zeta$  semi-circle. If the finger boundary is smooth, the solution  $f$  to the analytically continued equations (3.3) and (3.4) that match with the regular perturbation expansion as the physical region is approached must be unique and thus (5.1), (5.2) and (5.3) are the different behavior of the same global analytic function continued across the Stokes lines. Further, we note that  $g_1$  and  $g_2$  are both real on the imaginary axis in the interval  $(i q, \frac{i}{q})$  and therefore, the assumed finger symmetry about the channel centerline implies  $C_2^* = C_3$ . Thus, to the leading order, it is enough to require that each of (5.1) and (5.2)

hold in sectors I and II and in addition require

$$\text{Im } C_1 = 0 \quad (5.4)$$

Generally, for any  $\lambda$ , it is possible to find asymptotic solutions to (3.3) and (3.4) satisfying (5.1) and (5.2). However, (5.4) is violated. This implies that the tip is not smooth. Further, from the behavior  $\text{Im } f \sim \text{Im } C_1 g_1$  on the imaginary axis in the vicinity of  $\zeta = i$ , one is able to deduce that the singularity of  $f$  at  $\zeta = i$  is a branch cut corresponding to a cusp at the finger tip. At this point, it is appropriate to point out that this generalized solution with cusp is an asymptotic solutions to (3.3) and (3.4) for  $|\zeta| > 1$  in the 1st quadrant that matches with the perturbation series (2.25) and (2.26) as the physical region  $|\zeta| = 1$  is approached from the exterior. It is not an asymptotic solution to the original equations (2.12) and (2.13) on the entire arc of the entire semi-circle as a cusp would imply infinite curvature at the tip which is clearly not possible if (2.12) were to be satisfied at the tip as well. A solution to (3.3) and (3.4) is also a solution to (2.12) and (2.13) only when we require a unique analytic continuation of  $f$  across the semi-circular arc satisfying conditions (5.1), (5.2) and (5.3); which for a symmetric finger is equivalent to satisfying just (5.1) and (5.2) along with the requirement that on the imaginary  $\zeta$  axis there is some interval containing  $\zeta = i$  where  $\text{Im } f = 0$ . To the leading order, this latter condition is equivalent to (5.4). To find the actual expression for  $C_1$ , it is necessary to find and solve inner equations near  $\zeta = \frac{i}{g}$  where both the outer perturbation expansion and the WKB solutions  $g_1$  and  $g_2$  are invalid. In the following,  $\lambda$  together with  $Ca$  and  $\mathcal{G}$  will be treated as parameters and solutions found that satisfy (5.1) and (5.2). Once such solutions are found, the condition (5.4) will be imposed to see if they can be satisfied for some constraint relation between the three parameters.

The details of the inner region for  $\lambda < \frac{1}{2}$  depend on the assumptions made on  $Ca$ ,  $\mathcal{G}$  and  $\lambda$ . Here we will consider three different cases:

- (a)  $\frac{Ca \lambda^4}{(1-2\lambda)^2 \mathcal{G}} \ll 1$  or order unity,
- (b)  $(1-2\lambda)^2 \mathcal{G} \lambda^{-4} \ll Ca \ll \mathcal{G}^{1/2} (1-2\lambda) \lambda^{-2}$  and

(c)  $Ca \mathcal{G}^{-1/2} (1 - 2\lambda)^{-1} \lambda^2 \gg 1$  or order unity.

It will be assumed that in case (a),  $r \gg 1$ , in case (b)  $r \beta_1^{-3/2} \gg 1$  and in case (c),  $r \beta_2^{-21/8} \gg 1$ , where  $r$ ,  $\beta_1$  and  $\beta_2$  are as in (5.8), (5.11) and (5.12) below. However, as shall be seen later, a sufficient but perhaps not necessary condition for the above assumptions to be valid is that  $Ca = O(\mathcal{G}^{7/3})$  or larger in case (a) and  $Ca = O(\mathcal{G})$  in case (b). In case (c), the necessary assumption is equivalent to requiring that  $Ca^{-7/4} \lambda^{-7/2} (1 - 2\lambda)^{7/2} \gg 1$ . The solutions found in this case are found to satisfy this condition a posteriori provided  $Ca \gg \mathcal{G}$ . However, the possibility is open for new kind of solutions not described here with  $\lambda < 1/2$  but approaching  $\frac{1}{2}$  in the limit  $\mathcal{G} \rightarrow 0$  provided  $Ca = O(\mathcal{G})$  or smaller.

**5a. Case of  $\frac{Ca \lambda^4}{(1-2\lambda)^2 \mathcal{G}} \ll 1$  or order unity.**

In this case, it is appropriate to introduce inner variables

$$f = -\frac{2q^2}{1+q^2} F(\xi) r^{-4/7} \quad (5.5)$$

$$g = -\frac{2q^2}{1+q^2} \omega(\xi) r^{-4/7} \quad (5.6)$$

$$1 + iq\zeta = r^{-2/7} \xi \quad (5.7)$$

where

$$r = \frac{2^{3/2} q^3 (1 - q^2)^{1/2} 4}{(1 + q^2)^{3/2} \mathcal{G} \pi} \quad (5.8)$$

We will assume that  $(1 - 2\lambda)^{3/2} \lambda^{1/2} \gg \mathcal{G}$  so that  $r$  is indeed a large parameter. This assumption can become invalid in two ways: first, if  $\lambda$  is so small that  $\lambda \sim \mathcal{G}^2$ , and secondly when  $\lambda \sim \frac{1}{2}$  such that  $\frac{(1-2\lambda)}{\mathcal{G}^{3/2}} = O(1)$ . For the first case, since  $\lambda \gg \epsilon$ , i.e.  $\lambda \gg \mathcal{G}^{1/2} Ca^{1/2}$  for the SPHR boundary conditions to be valid, it is clear that the assumption can possibly be violated only when  $\mathcal{G}^{1/2} Ca^{1/2} \ll \mathcal{G}^2$ , i.e. when  $Ca \ll \mathcal{G}^3$ . However, a glance at the original equations (1) and (2) would seem to suggest that under these conditions this film effects can be neglected and that the MS boundary conditions are valid. We will assume this is indeed the case. Earlier work on the

MS boundary conditions rule out any possibility of  $\lambda < 1/2$ . For the second case when  $\lambda \sim \frac{1}{2}$ , the assumptions made in this case imply that  $(1 - 2\lambda) = O(Ca^{1/2} \mathcal{G}^{-1/2})$  or larger. Thus the assumption of  $r$  large can be violated only if  $Ca \ll \mathcal{G}^{7/3}$ . Thus if  $Ca = O(\mathcal{G}^{7/3})$  or larger, the assumption of large  $r$  follows. However, this may not be a necessary assumption. When  $r$  is not large because  $q$  is small, one can use the scaling in section 6 with  $p$  replaced by  $q$ ; however the analysis for such a case is not presented here.

From the transformations (5.5) and (5.6), to the leading order in  $r^{-2/7}$ , we find that (3.3) and (3.4) reduces to

$$F + \omega = -\beta_1 \frac{1}{(\xi - F')^{1/3}} + \frac{(F'' - 1)}{(\xi - F')^{3/2}} \quad (5.9)$$

and

$$\omega' = \beta_2 \frac{1}{(\xi - F')^{1/3}} + \beta_3 \frac{(F'' - 1)}{(\xi - F')^{11/6}} \quad (5.10)$$

where

$$\beta_1 = \frac{3.878}{2\sqrt{3}} \left( \frac{Ca}{\mathcal{G}} \right)^{1/6} \frac{(1 + q^2)^{4/3} (1 - q^2)^{1/3}}{q^{8/3} r^{-2/3} \mathcal{G}^{-2/3}} \quad (5.11)$$

$$\beta_2 = 1.3375 \frac{Ca^{2/3}}{\mathcal{G}^{8/21}} \frac{(1 - q^4)^{4/3}}{4q^{8/3}} (r \mathcal{G})^{8/21} \quad (5.12)$$

and

$$\beta_3 = 1.3375 \sqrt{\frac{3}{2}} \frac{\pi}{32} \frac{Ca^{7/6}}{\mathcal{G}^{13/42}} (1 - q^4)^{5/6} \frac{(1 + q^2)}{q^{11/3}} (r \mathcal{G})^{17/21} \quad (5.13)$$

Eliminating  $\omega$  between (5.9) and (5.10), we obtain

$$F' + \frac{\beta_2}{(\xi - F')^{1/3}} + \beta_3 \frac{(F'' - 1)}{(\xi - F')^{11/6}} = \frac{\beta_1 (1 - F'')}{3(\xi - F')^{4/3}} + \frac{F'''}{(\xi - F')^{3/2}} + \frac{3}{2} \frac{(F'' - 1)^2}{(\xi - F')^{5/2}} \quad (5.14)$$

Note that in order to obtain (5.9) and (5.10) from (3.3) and (3.4), we have to assume  $Ca \ll \mathcal{G}^{1/7} \lambda^{-4/7} (1 - 2\lambda)^{2/7}$  or otherwise the product  $Ca n_*(\zeta)$  for  $\xi = O(1)$  is not small making the Bretherton approximation (eqns 2.3-2.6) invalid.

Further, in this case  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  are far smaller than unity and one can neglect to the leading order all terms in (5.14) involving  $\beta_1$ ,  $\beta_2$  and  $\beta_3$ . Going back to (5.9)

and (5.10), this means that to the leading order  $\omega = 0$  without any loss of generality and

$$F = \frac{(F'' - 1)}{(\xi - F')^{3/2}} \quad (5.15)$$

which is the same equation as for the MS case when thin films and the variation of transverse curvature are totally ignored. The solution  $F$  to (5.15) matches with the outer solution in sectors I and II provided we require that the  $F$  goes to zero as  $\xi \rightarrow \infty$  for  $Arg \xi$  in the interval  $[0, 6\pi/7)$ . This requirement gives a unique solution to (5.15) and thus matching with (5.1) and (5.2) for  $|1+iq\zeta| \ll 1$ , one finds  $C_1$  and  $C_2$ . The details of this case have been discussed in the context of the analytic theory of velocity selection of a bubble using the MS boundary conditions (Tanveer, 1989a) and also in the context of dendritic crystal growth (Tanveer, 1989b), where the same equation (5.15) appears.  $Im C_1$  can only be zero if the solution to (5.15) satisfying the above stated conditions is real on the real axis and this is found not to be the case. Thus there are no solutions in this case.

When  $Ca = O(\mathcal{G} (1 - 2\lambda)^2 \lambda^{-4})$ ,  $\beta_1 = O(1)$ , while other parameters  $\beta_2$  and  $\beta_3$  are small. Thus to the leading order, one can ignore the  $\beta_2$  and  $\beta_3$  terms in (5.14). This means that in this situation, we can ignore the thin film effect on the kinematic equation but include it in the dynamic conditions. Then going back to (5.9), the relevant equation in this case to the leading order is

$$F = -\beta_1 \frac{1}{(\xi - F')^{1/3}} + \frac{(F'' - 1)}{(\xi - F')^{3/2}} \quad (5.16)$$

Numerical solutions of this equation for  $\beta_1$  in the range  $(0, 10)$  were sought and a unique solution to (5.16) was found that tends to 0 as  $\xi \rightarrow \infty$  for  $Arg \xi$  in the interval  $[0, 6\pi/7)$  as necessary to match with (5.1) and (5.2). However, such a solution was not real on the real  $\xi$  axis as would have to be the case to satisfy (5.4). The numerical method is a trivial modification of the earlier procedure (Tanveer, 1989b) for finding appropriate solutions to (5.15). Indeed with increase in  $\beta_1$ , the residual of the numerically imposed

condition for a smooth tip increased with increasing  $\beta_1$  suggesting that an increase in  $\beta_1$  causes larger mismatch in the tip slope between the two sides of the cusp.

**5b. Case of  $(1 - 2\lambda)^2 \mathcal{G} \lambda^{-4} \ll Ca \ll \mathcal{G}^{1/2} (1 - 2\lambda) \lambda^{-2}$**

In this case,  $\beta_1$  is large.  $\beta_3$  is small; but  $\beta_2$  can be large or small compared to unity. It is appropriate to introduce rescaled variables

$$\xi_1 = \beta_1^{-3/7} r^{2/7} (1 + i q \zeta) \quad (5.17)$$

$$F_1(\xi_1) = -\beta_1^{-6/7} \frac{(1 + q^2)}{2q^2} r^{4/7} f(\zeta) \quad (5.18)$$

$$\omega_1(\xi_1) = -\beta_1^{-6/7} \frac{(1 + q^2)}{2q^2} r^{4/7} g(\zeta) \quad (5.19)$$

where  $r$  and  $\beta_1$  are as defined earlier in (5.8) and (5.11).

We will assume that  $r \beta_1^{-3/2} \gg 1$ , implying  $Ca^{-1/4} \mathcal{G}^{-3/4} \lambda^{-1/2} (1 - 2\lambda)^2 \gg 1$ . The only way this assumption can be violated is if  $\lambda$  is very close to  $\frac{1}{2}$ . However, in this case, when  $\lambda \sim \frac{1}{2}$ ,  $(1 - 2\lambda) \gg Ca \mathcal{G}^{-1/2}$ . Thus a sufficient condition for the validity of the assumption is that  $Ca \gg O(\mathcal{G})$  or larger. Note that the Bretherton approximation is valid, i.e.  $Ca n_x \ll 1$  for  $\xi_1 = O(1)$ , since  $Ca n_x = O(Ca^{27/28} \mathcal{G}^{-3/28} \lambda^{3/7} (1 - 2\lambda)^{-3/14}) \ll 1$ . This is true because  $(1 - 2\lambda) \gg \frac{Ca \lambda^2}{\mathcal{G}^{1/2}}$  and  $Ca^{3/4} \ll 1$ .

Then to the leading order in  $\beta_2^{3/7} r^{-2/7}$ , for  $\xi_1 = O(1)$ , equations (3.3) and (3.4) become

$$\omega_1 + F_1 = -\frac{1}{(\xi_1 - F'_1)^{1/3}} + \beta_1^{-3/2} \frac{(F''_1 - 1)}{(\xi_1 - F'_1)^{3/2}} \quad (5.20)$$

$$\omega'_1 = \frac{\beta_2}{\beta_1^{4/7}} \frac{1}{(\xi_1 - F'_1)^{1/3}} + \frac{\beta_3}{\beta_1^{17/14}} \frac{(F''_1 - 1)}{(\xi_1 - F'_1)^{11/6}} \quad (5.21)$$

From the definition of  $\beta_1$ ,  $\beta_2$  and  $\beta_3$ , it is clear that each of the quantities  $\beta_1^{-3/2}$ ,  $\frac{\beta_2}{\beta_1^{4/7}}$  and  $\frac{\beta_3}{\beta_1^{17/14}}$  are much smaller than one. Thus to the leading order, the appropriate equation is

$$F_{1_0} = -\frac{1}{(\xi_1 - F'_{1_0})^{1/3}}. \quad (5.22)$$

For large  $\xi_1$ , the solution must match with  $f \sim \tilde{f}_1$ . On examination of (5.22) the behavior of  $F_1(\xi_1)$  needed to assure this matching is: This implies that for large  $\xi_1$ , the asymptotic behavior of  $F_{1_0}$  must be

$$F_{1_0} \sim -\frac{1}{\xi_1^{1/3}} \quad (5.23)$$

On linearizing (5.22) about this behavior, we obtain the homogenous equation

$$F'_H + 3 \xi_1^{4/3} \left[ 1 - \frac{4}{9 \xi_1^{7/3}} \right] F_H = 0 \quad (5.24)$$

The solution to this is of the form

$$F_{H_1} = \xi_1^{4/3} e^{-\frac{6}{7} \xi_1^{7/3}} \quad (5.25)$$

This is transcendentally small for  $Arg \xi_1$  in  $(-3\pi/14, 3\pi/14)$  and transcendentally large for  $Arg \xi_1$  in  $(3\pi/14, 9\pi/14)$ . A second possible transcendental behavior for large  $\xi_1$  is found by retaining the highest derivative term in (5.20). Linearizing about (5.23), and examining the homogeneous equation it is easy to see that a possible transcendental behavior for large  $\xi$  is

$$F_{H_2} = e^{\beta_1^{3/2} \frac{2}{7} \xi_1^{7/6} [1 + o(1)]} \quad (5.26)$$

and this is transcendentally large for  $Arg \xi_1$  in the interval  $(-3\pi/7, 3\pi/7)$  and transcendentally small in  $(3\pi/7, 9\pi/7)$ . However, none of the transcendental behavior in (5.25) or (5.26) match with multiples of  $g_1$  and  $g_2$  directly. This is because there is an intermediate scale when  $|1 + iq\zeta| = Ca^{2/7} \frac{\lambda}{(1-2\lambda)}$  where the effect of nonzero  $Ca$  enters into the WKB solutions (4.13) and (4.14) and the further approximation  $g_1$  and  $g_2$  as given by (4.17) are invalid. One can obtain the correct asymptotic behavior by going back to (4.13) and (4.14). It is easier in our case to directly obtain the same results from equation (5.20) and (5.21). By linearizing about  $F_1 = 0$ , which is valid for large enough  $\xi_1$ , we obtain a pair of linear differential equations. The homogeneous part of those equations generate transcendentally small terms. By introducing rescaled variables

$$\xi_1 = \beta_1^{9/7} \xi_3 \quad (5.27)$$

into the resulting equations, they can be reduced to one homogeneous equation for the homogeneous solution  $\tilde{F}_H(\xi_3)$ . For  $\xi_3 = O(1)$  and  $\beta_1 \gg 1$ , this equation reduces to:

$$\frac{d^3 \tilde{F}_H}{d\xi_3^3} - \frac{1}{3} \beta_1^3 \xi_3^{1/6} [1 + o(1)] \frac{d^2 \tilde{F}_H}{d\xi_3^2} - \beta_1^6 \xi_3^{3/2} [1 + o(1)] \frac{d\tilde{F}_H}{d\xi_3} = 0 \quad (5.28)$$

The two independent WKB approximations to the independent solutions are:

$$\tilde{F}_{H_1} = e^{\frac{1}{5}\beta_1^3 \left[ \int_0^{\xi_3} d\xi_3 \left[ \xi_3^{1/6} - \sqrt{\xi_3^{1/3} + 36\xi_3^{3/2}} \right] \right]} + o(1) \quad (5.29)$$

$$\tilde{F}_{H_2} = e^{\frac{1}{5}\beta_1^3 \left[ \int_0^{\xi_3} d\xi_3 \left[ \xi_3^{1/6} + \sqrt{\xi_3^{1/3} + 36\xi_3^{3/2}} \right] \right]} + o(1) \quad (5.30)$$

For large  $\xi_3$ , the behavior in (5.29) and (5.30) matches with  $g_1$  and  $g_2$  as given by (4.17) for small  $|1 + i q\zeta|$ . On the otherhand, for small  $\xi_3$ , the behavior matches with  $F'_{H_1}$  and  $F'_{H_2}$  as given by (5.25) and (5.26) respectively. The  $o(1)$  in the expressions (5.29) and (5.30) are not important in our analysis other than that they are real for real  $\xi_3$ .

Including the leading order transcendentally small correction term, the asymptotic behavior of solution  $F_1$  for  $Arg \xi_1$  in the interval  $[0, 3\pi/7)$  is

$$F_1 \sim -\frac{1}{\xi_1^{1/3}} + \dots + A_1 F_{H_1} \quad (5.31)$$

On the other hand for  $Arg \xi_1$  in  $(3\pi/7, 9\pi/7)$ , the transcendental correction cannot include a multiple of  $F_{H_1}$  which gets large in this sector as  $\xi_1 \rightarrow \infty$  and does not match with a multiple of (5.30) as it must in order to match with the behavior (5.2) in sector II of the  $\zeta$  plane in Fig. 3.

As far as the differential equation (5.22), a unique solution is obtained by numerically integrating the solution from  $\xi_1 = \bar{L}e^{i3\pi/7}$  to  $\xi_1 = \bar{\xi}_1$  where  $\bar{L}$  is chosen to be 12 and  $\bar{\xi}_1 = 1$  with the condition (5.23) imposed as an initial condition. The choice of the initial point and initial condition was made to suppress  $F_{H_1}$  where it is transcendentally large at  $\xi_1 = \bar{\xi}_1$  for large  $\bar{L}$ . The solution was found to have a non zero imaginary part

at  $\bar{\xi}_1$  implying that the coefficient of  $F_{H_1}$  is not purely real implying that  $Im C_1$  in (5.1) is not zero. Thus no finger solution with smooth tip exists in this case. The results were checked by varying  $\bar{L}$  and  $\bar{\xi}_1$ .

The leading order asymptotic solution  $F_{1_0}$  is not valid for all  $\xi_1$ . Its derivative is singular at some  $\xi_1 = \xi_{1_0}$  in the upper half  $\xi_1$  plane where the local singularity behavior can be directly deduced from (5.22) to be

$$F'_{1_0} \sim -\left(\frac{1}{4}\right)^{3/4} (\xi_1 - \xi_{1_0})^{-3/4} \quad (5.32)$$

Thus the approximation  $F_1 \sim F_{1_0}$  becomes invalid near  $\xi = \xi_{1_0}$  as the next order approximation  $F_{1_1}$  becomes singular owing to inclusion of the  $\beta_1^{-3/2}$  term in (5.20). However, this does not affect the conclusion about the non-existence of solution. Here, we present the details of the analysis of this innermost region for the sake of completeness. We rescale variables once again in the neighborhood of  $\xi_1 = \xi_{1_0}$ :

$$\xi_1 = \xi_{1_0} + \beta_1^{-12/7} \xi_2 \quad (5.33)$$

$$F_1 = -\beta_1^{-3/7} F_2(\xi_2) \quad (5.34)$$

Then for  $\xi_2 = O(1)$ , to the leading order (5.20) and (5.2') simplify to

$$F_2 = \frac{1}{F_2'^{1/3}} + \frac{F_2''}{F_2'^{7/2}} \quad (5.35)$$

For large  $\xi_2$ , it is clear that the asymptotic behavior that is consistent with (5.32) is given by

$$F_2 = \left(\frac{1}{4}\right)^{3/4} \xi_2^{-3/4} \quad (5.36)$$

The transcendental correction to this behavior found by linearizing (5.35) about (5.36) and examining the homogeneous part of the equation is a multiple of

$$e^{\frac{5}{21}\left(\frac{1}{4}\right)^{1/8} \xi_2^{7/8}} [1 + o(1)] \quad (5.37)$$

This is transcendentally large for  $\text{Arg } \xi_2$  in  $(-4\pi/7, 4\pi/7)$  and transcendentally small for  $\text{Arg } \xi_2$  in  $(4\pi/7, 12\pi/7)$ . It is necessary to suppress the transcendentally large behavior for large  $\xi_2$  when  $\text{Arg } \xi_2$  is in  $(-4\pi/7, 4\pi/7)$  since the transcendental behavior given in the above equation matches to the following transcendentally small correction for small  $\beta_1$ :

$$e^{\frac{1}{3}\beta_1^{3/2} \int_{\xi_{1,0}}^{\xi_1} (\xi_1 - F'_{1,0})^{1/6} d\xi_1} [1 + o(1)] \quad (5.38)$$

This was determined by linearizing (5.20) and (5.21) about  $F_{1,0}$  and looking for WKB type solutions for small  $\beta_1$  to the resulting homogeneous equation. The expression in (5.38) matches to a constant multiple of (5.30) for large  $\xi_1$  and small  $\xi_3$ . On the other hand, for large  $\xi_3$  and small  $|1 + iq\zeta|$ , a multiple of (5.30) will match with  $C_2 g_2$  in sector 2 in Fig. 3 as required. Since (5.35) is autonomous, the other degree of freedom of the solution in the second order differential equation is the arbitrariness of the choice of origin of  $\xi_2$ , but this does not affect the matching. Thus a unique value of  $C_2$  in (5.2) can be determined in this matching procedure. However, as we concluded before, the finger tip in this case is not smooth.

**3c. Case of  $Ca G^{-1/2} (1 - 2\lambda)^{-1} \lambda^2 \gg 1$  or order unity.**

In this case, each of  $\beta_2, \beta_1$  are large, but  $\beta_3$  can be large or small or  $O(1)$  compared to unity. It is appropriate to introduce the transformation

$$\xi_1 = \beta_2^{-3/4} r^{2/7} (1 + iq\zeta) \quad (5.39)$$

$$F_1(\xi_1) = -\beta_2^{-3/2} r^{4/7} \frac{(1 + q^2)}{2q^2} f(\zeta) \quad (5.40a)$$

$$\omega_1(\xi_1) = -\beta_2^{-3/2} r^{4/7} \frac{(1 + q^2)}{2q^2} g(\zeta) \quad (5.40b)$$

into equations (3.3) and (3.4), which to the leading order in  $\beta_1^{3/4} r^{-2/7}$ , assumed small, for  $\xi_1 = O(1)$ , on elimination of  $\omega_1$ , reduces to

$$F'_1 + \frac{1}{(\xi_1 - F'_1)^{1/3}} + \frac{\beta_3}{\beta_2^{17/8}} \frac{(F'_1 - 1)}{(\xi_1 - F'_1)^{11/6}} = \frac{\beta_1}{\beta_2^{7/4}} \frac{(1 - F'_1)}{3(\xi_1 - F'_1)^{4/3}}$$

$$+ \frac{1}{\beta_2^{21/8}} \left[ \frac{F_1'''}{(\xi_1 - F_1')^{3/2}} + \frac{3}{2} \frac{(F_1'' - 1)^2}{(\xi_1 - F_1')^{5/2}} \right] \quad (5.41)$$

Here, in this case we assume that  $Ca^{-1/2} \lambda^{-1} (1 - 2\lambda) \gg 1$  in order that  $r^{-2/7} \beta_2^{3/4} \gg 1$ . This assumption is found to be satisfied for the class of solutions found for  $Ca \gg G$ . This will be assumed to be the case. Note that for  $\xi_1 = O(1)$ ,  $Ca n_* = O(\lambda^{1/2} (1 - 2\lambda)^{-1/2} r^{1/7} Ca \beta_2^{-3/8})$  and using the definition of  $r$  and  $\beta_2$ , this is found to  $O(Ca^{3/4}) \ll 1$ . Thus the Bretherton approximation used in the derivation of (5.41) from (3.3) and (3.4) is indeed valid.

First, when  $Ca G^{-1/2} (1 - 2\lambda)^{-1} \lambda^2 \gg 1$ , it is clear that  $\beta_3 \beta_2^{-17/8} \ll 1$ ,  $\beta_1 \beta_2^{-7/4} \ll 1$  and  $\beta_2^{-21/8} \ll 1$ . Of these small parameters  $\beta_1 \beta_2^{-7/4}$  is the largest. To the leading order

$$F_1' = F_{1_0}' \quad (5.42)$$

where  $F_{1_0}'$  is determined by the nonlinear algebraic equation

$$F_{1_0}' + \frac{1}{(\xi_1 - F_{1_0}')^{1/3}} = 0 \quad (5.43)$$

The next order correction is clearly

$$F_{1_1}' = \frac{\beta_1}{\beta_2^{7/4}} \frac{(1 - F_{1_0}'')}{3(\xi_1 - F_{1_0}')^{4/3} + 1} \quad (5.44)$$

Note that in the upper half  $\xi_1$  plane,  $F_{1_1}'$  is singular when

$$\xi_1 = 4e^{i3\pi/4} 3^{-3/4} \equiv \xi_{1_0} \quad (5.45)$$

where the local behavior of  $F_{1_0}'$  and  $F_{1_1}'$  are given by

$$F_{1_0}' \sim \frac{3}{4} \xi_{1_0} - \left(\frac{3}{8} \xi_{1_0}\right)^{1/2} (\xi_1 - \xi_{1_0})^{1/2} + \dots \quad (5.46)$$

$$F_{1_1}' \sim \frac{\beta_1}{\beta_2^{7/4}} \frac{1}{8} \frac{1}{\left(\frac{1}{4} \xi_{1_0}\right)^{1/3}} \frac{1}{\xi_1 - \xi_{1_0}} \quad (5.47)$$

as can be seen by analyzing (5.43) and (5.44). The minus sign preceding the  $\left(\frac{3}{8} \xi_{1_0}\right)^{1/2}$  term in (5.46) is the only choice consistent with the behavior  $F_{1_0}' \sim -\xi_1^{-1/3}$  for large  $\xi_1$  on

the real positive  $\xi_1$  axis (with the principal branch choice) as was determined numerically by solving the nonlinear algebraic equation (5.43) for  $F'_{1_0}$  for each  $\zeta$ , taking care that there is no large jump in the the argument of  $\xi_1 - F'_{1_0}$  between neighboring  $\zeta$  points (assuring us that we are on the same branch of the Riemann sheet). Now, it is possible to continue to try to find the next order correction  $F_{1_2}$  in a similar fashion. However, for our purpose it will suffice to note that for  $Ca \gg \mathcal{G}^{1/2} (1 - 2\lambda) \lambda^{-2}$ , the two term leading order behavior of  $F_1$  is:

$$F_1 \sim F_{1_0} + F_{1_1} \equiv F_A \quad (5.48)$$

To find transcendentally small corrections to this behavior, we linearize equation (5.41) about  $F_A$  and look at the homogeneous part of the equation whose solution will generate terms with transcendental dependence in  $\beta_1$ ,  $\beta_2$  and  $\beta_3$ . By a standard dominant balance argument on the resulting linear equation, tedious, but routine calculation shows two possible behaviors for the solution to the homogeneous equation. These will be denoted by  $F_{H_1}$  and  $F_{H_2}$  and their derivatives are given by

$$F'_{H_1} = (\xi_1 - F'_{1_0})^{4/3} e^{-\frac{\beta_2^{7/4}}{\beta_1} \left[ \int_0^{\xi_1} d\xi_1 \left[ (\xi_1 - F'_{1_0})^{4/3} + \frac{1}{3} \right] + o(1) \right] + \int_0^{\xi_1} d\xi_1 \frac{4(\xi_1 - F'_{1_0})^{1/3} (1 - F''_{1_0})}{13(\xi_1 - F'_{1_0})^{4/3} + 1}} \quad (5.49)$$

$$F'_{H_2} = e^{\frac{1}{3}\beta_1\beta_2^{7/6} \left[ \int_0^{\xi_1} (\xi_1 - F'_{1_0})^{1/6} d\xi_1 + o(1) \right]} \quad (5.50)$$

Note that in (5.49), the occurrence of the second term in the exponent is due to the inclusion of  $F'_{1_1}$  as given by (5.47). The product of  $\frac{\beta_2^{7/4}}{\beta_1}$  and  $o(1)$  term appearing in the exponent in (5.49) depends on the coefficients  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  and terms of this product can in some cases be larger than unity and therefore exceed the second term in the exponent in (5.49) which does not contain any parameter. However, it is convenient in our case to isolate the parameter independent  $O(1)$  term in the exponent in (5.49), as we have done, and write the remainder which has no parameter independent  $O(1)$  term as the product of  $\frac{\beta_2^{7/4}}{\beta_1}$  and  $o(1)$ . For our purpose in determining the leading order

selection rule, it will not be necessary to find any of the  $o(1)$  terms in the exponents in (5.49) and (5.50) aside from noting that they are real on the real  $\xi_1$  axis. To connect the behavior the behavior of transcendental terms in (5.49) and (5.50) to the far field behavior where transcendental terms are linear combinations of  $g_1$  and  $g_2$  as defined in (4.17), one needs to introduce an intermediate scale variable:

$$\xi_3 = \beta_1^{-12/7} \xi = \beta_1^{-12/7} \beta_2^{3/4} \xi_1 \quad (5.51)$$

$$\tilde{F}_H(\xi_3) = F_H(\xi_2) \quad (5.52)$$

Then for  $\xi_3 = O(1)$ ,  $\beta_1 \rightarrow \infty$ , one finds the equation for  $\tilde{F}_H$  reduces to

$$\frac{d^3 \tilde{F}_H}{d\xi_3^3} - \frac{1}{3} \beta_1^3 \xi_3^{1/6} [1 + o(1)] \frac{d^2 \tilde{F}_H}{d\xi_3^2} - \beta_1^6 \xi_3^{3/2} [1 + o(1)] \frac{d\tilde{F}_H}{d\xi_3} = 0 \quad (5.53)$$

The two possible solutions to this equation for large  $\beta_1$  are

$$\tilde{F}'_{H_1} = e^{\frac{1}{6}\beta_1^3 \left[ \int_0^{\xi_3} d\xi_3 \left[ \xi_3^{1/6} - \sqrt{\xi_3^{1/3} + 36\xi_3^{3/2}} \right] + o(1) \right]} \quad (5.54)$$

$$\tilde{F}'_{H_2} = e^{\frac{1}{6}\beta_1^3 \left[ \int_0^{\xi_3} d\xi_3 \left[ \xi_3^{1/6} + \sqrt{\xi_3^{1/3} + 36\xi_3^{3/2}} \right] + o(1) \right]} \quad (5.55)$$

For large  $\xi_3$ , the behavior in (5.54) and (5.55) matches with  $g'_1$  and  $g'_2$  respectively. On the other hand for small  $\xi_3$ , it is clear that (5.54) and (5.55) correspond to  $F'_{H_1}$  and  $F'_{H_2}$  as given by (5.49) and (5.50) respectively. The details of the Stokes lines in different nested inner regions  $\xi_1 = O(1)$  and  $\xi_3 = O(1)$  are shown in Figs. 5 and 6. The solid lines are the Stokes lines with respect to  $F'_{H_1}$  and the dotted lines are the ones respect to  $F'_{H_2}$ . Thus, in order that there be no transcendently large term in sector I and II of Fig. 3, it is necessary that the form of transcendental of transcendental correction to (5.48) for large  $\xi_1$  be given by a multiple of  $F'_{H_1}$  in sector I and of  $F'_{H_2}$  in sector II of Fig. 5. Also, note that  $g_1$  is completely real for  $\zeta$  on the imaginary axis between  $iq$  and  $i/q$ . Also,  $\tilde{F}'_{H_1}$  is real on the real and positive  $\xi_3$  axis. Thus the requirement that the imaginary part of the transcendental correction for  $f$  be completely real on some interval

on the imaginary  $\zeta$  axis including  $i$ , to the leading order, is equivalent to requiring that the multiple of  $g_1$  in the transcendental correction in sector I of Fig. 3 be real, which in turn implies that the coefficient of  $F'_{H_1}$  in sector I of Fig. 5 be real. Now, to find what this coefficient must be, we introduce an inner variable in the neighborhood of  $\xi_1 = \xi_{1_0}$ , where the approximation  $F' \sim F'_{1_0}$  becomes invalid.

$$\xi_1 = \xi_{1_0} + \left( \frac{\beta_1}{\beta_2^{7/4}} \right)^{2/3} \left( \frac{1}{4} \xi_{1_0} \right)^{1/3} \left( \frac{3}{2} \right)^{1/3} \xi_2 \quad (5.56)$$

$$F'_1 = \frac{3}{4} \xi_{1_0} + \left( \frac{\beta_1}{\beta_2^{7/4}} \right)^{1/3} \left( \frac{1}{4} \xi_{1_0} \right)^{2/3} \left( \frac{3}{2} \right)^{2/3} F'_2(\xi_2) \quad (5.57)$$

Then for  $\xi_2 = O(1)$ , to the leading order (5.41) reduces to:

$$-\xi_2 + F_2'^2 = -F_2'' + \beta_4 F_2''' \quad (5.58)$$

where

$$\beta_4 = 3 \left( \frac{2}{3} \right)^{1/3} \left( \frac{1}{4} \xi_{1_0} \right)^{-1/2} \beta_1^{-5/3} \beta_2^{7/24} \quad (5.59)$$

Under the conditions of this case, we note that  $\beta_4 \ll 1$ . To the leading order, if we ignore the  $\beta_4$  term completely, we get the exact solution

$$F_2' = \frac{Ai'(\xi_2)}{Ai(\xi_2)} \equiv F_{2_0}' \quad (5.60)$$

and this matches with (5.46) for  $|\xi_2| \gg 1$  for argument  $\xi_2$  in the interval  $(-\pi, \pi/3)$  corresponding to small  $|\xi_1 - \xi_{1_0}|$  with  $Arg(\xi_1 - \xi_{1_0})$  in the range  $(-3\pi/4, 7\pi/12)$  as can be seen from the asymptotic expansion of the Airy function and properties of its analytic continuation in different sectors over different ranges of arguments. To the behavior (5.60), one can add a transcendently small correction for small  $\beta_4$  or large  $\xi_2$  that is a multiple of  $e^{\beta_4^{-1}(\xi_2 - \xi_{2_j})}$  away from the immediate neighborhood of the set of  $\xi_{2_j}$ , the  $j$ th zero of  $Ai(\xi_2)$ ,  $j$  ranging from 1 to  $\infty$ . For large  $\xi_2$  this correction is transcendently small only when  $Arg \beta_4^{-1} \xi_2$  is in the range  $(\pi/2, 3\pi/2)$ , i.e.  $Arg(\xi_1 - \xi_{1_0})$  in  $(3\pi/8, 11\pi/8)$ . This transcendental term matches with  $F_{H_2}$ . The multiplicative constant term is found

by requiring that the solution to (5.58) contain no transcendently large term in  $\xi_2$  for  $Arg \xi_2$  in the interval  $(-\pi/8, \pi/8)$ . However, we will not need this constant multiple in the determination of the finger width. The transcendental correction for large  $\xi_2$  with the argument of  $\xi_2$  in the interval  $(-\pi, -\pi/3)$ , i.e.  $Arg(\xi_1 - \xi_{1_0})$  in  $(-3\pi/4, -\pi/12)$  can be directly calculated from the behavior (5.60). We find that for  $Arg \xi_2$  in  $(-\pi, -\pi/3)$ ,

$$F'_2 \sim -\xi_2^{1/2} - i \xi_2^{1/2} e^{\frac{4}{3}\xi_2^{3/2}} \quad (5.61)$$

and this matches with the behavior

$$F'_1 = F'_{1_0} + A_1 F'_{H_1} \quad (5.62)$$

provided that

$$A_1 = i u_1^{-1/2} u_2 e^{3\beta_2^{7/4} \beta_1^{-1} \left[ \int_0^{\xi_{1_0}} [(\xi_1 - F'_{1_0})^{-1/3} + \frac{1}{3}] + o(1) \right] d\xi_1} e^{\int_0^{\xi_{1_0}} d\xi_1 \left[ \frac{4(\xi_1 - F'_{1_0})^{1/3} (1 - F''_{1_0})}{[3(\xi_1 - F'_{1_0})^{4/3} + 1]} - \frac{1}{2(\xi_1 - \xi_{1_0})} \right] - \frac{1}{2} \ln(-\xi_{1_0})} \quad (5.62)$$

where

$$u_1 = \left( \frac{\beta_1}{\beta_2^{7/4}} \right)^{2/3} \left( \frac{1}{4} \xi_{1_0} \right)^{1/3} \left( \frac{3}{2} \right)^{1/3} \quad (5.64)$$

$$u_2 = \left( \frac{\beta_1}{\beta_2^{7/4}} \right)^{1/3} \left( \frac{1}{4} \xi_{1_0} \right)^{2/3} \left( \frac{3}{2} \right)^{2/3} \quad (5.65)$$

On numerical evaluation, we found

$$Im \int_0^{\xi_{1_0}} d\xi_1 \left[ (\xi_1 - F'_{1_0})^{4/3} + \frac{1}{3} \right] = -0.473 \quad (5.66)$$

$$Im \left[ \int_0^{\xi_{1_0}} d\xi_1 \left[ \frac{4(\xi_1 - F'_{1_0})^{1/3} (1 - F''_{1_0})}{[3(\xi_1 - F'_{1_0})^{4/3} + 1]} - \frac{1}{2(\xi_1 - \xi_{1_0})} \right] - \frac{1}{2} \ln(-\xi_{1_0}) \right] = -1.27 \quad (5.67)$$

The condition that  $A_1$  be real is then equivalent to

$$0.473 \frac{3\beta_2^{7/4}}{\beta_1} [1 + o(1)] - 1.27 + \frac{1}{2} Arg u_1 = -\frac{\pi}{2} + Arg u_2 + n \pi \quad (5.68)$$

Thus, if we drop the  $o(1)$  term in (5.68), we obtain

$$3 \frac{\beta_2^{7/4}}{\beta_1} \sim \frac{(n \pi - \frac{\pi}{8} + 1.27)}{0.473} \quad (5.69)$$

Where  $n$  is a positive integer. Then going back to the assumption made in this case,  $n$  must be a large integer. Generally, there is not much point in including the term  $-\pi/8 + 1.27$  in the right hand side of (5.68) because the product of  $\beta_2^{7/4}/\beta_1$  and  $o(1)$  in (5.68) that is neglected in (5.69) is larger than unity. However, we still include it to make the formula more accurate in the special case when this product is smaller than unity as is the case when each of the conditions  $\frac{\beta_2 \beta_2^{7/4}}{\beta_1^{17/8} \beta_1} \ll 1$  and  $\beta_2^{-21/8} \beta_2^{7/4} / \beta_1 \ll 1$ , i.e. for large but not too large  $n$ . Equation (5.69) implies that for  $\frac{\lambda^2}{(1-2\lambda)} \frac{Ca}{\rho^{1/2}} \gg 1$ , for sufficiently small  $Ca$ ,

$$\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} \sim 2.06 (n + 0.279) \frac{G^{1/2}}{Ca} \quad (5.70)$$

The above expression is only valid for large  $n$ , i.e. higher branches of solution since for small  $n$ , this would imply  $\frac{\lambda^2}{(1-2\lambda)} \frac{Ca}{\rho^{1/2}} = O(1)$ .

For the lower branches, i.e. when  $n$  is order unity,  $\frac{\beta_2}{\beta_1^{7/4}} = O(1)$ , implying  $Ca G^{-1/2} (1-2\lambda)^{-1} \lambda^2 = O(1)$  one has to resort to numerical calculations of the solution to the differential equation

$$F_1' + \frac{1}{(\xi_1 - F_1')^{1/3}} = \frac{\beta_1}{3\beta_2^{7/4}} \left[ \frac{1 - F_1''}{(\xi_1 - F_1')^{4/3}} \right] \quad (5.71)$$

with the requirement that the behavior of  $F_1'$  for large  $\xi_1$  with  $Arg \xi_1$  in the interval  $(-\pi/12, 7\pi/12)$  be given by

$$F_1' \sim -\frac{1}{\xi_1^{1/3}} + \frac{\beta_1}{3\beta_2^{7/4}} \frac{1}{\xi_1^{4/3}} \quad (5.72)$$

and that it be real on the positive real  $\xi_1$  axis for sufficiently large  $\xi_1$ . Such solutions were found numerically by integrating the first order equation (5.71) for  $F_1'$  from  $L e^{i\pi/2}$  to  $\xi_1 = \bar{\xi}_1$ , where we chose  $L = 12$  and  $\bar{\xi}_1 = 0$ . with the asymptotic condition (5.72)

satisfied at the far end point. We required that  $Im F_1'(\bar{\xi}_1) = 0$  in order to determine  $\frac{\beta_1}{3\beta_2^{7/4}}$ . We doubled  $L$  and changed  $\bar{\xi}_1$  to other real positive values without noticing any change in the final results to the accuracy quoted. We found the first two values of  $\frac{\beta_1}{3\beta_2^{7/4}}$  to be 9.848 and 15.69 implying

$$\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} = 3.061 \frac{\mathcal{G}^{1/2}}{Ca} \quad (5.73)$$

for the first branch compared to

$$\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} = 4.878 \frac{\mathcal{G}^{1/2}}{Ca} \quad (5.74)$$

for the second branch. If we were to base the prediction on the formula (5.70), then for  $n = 1$  and 2, the coefficients on the right hand sides of (5.73) and (5.74) would have been 2.634 and 4.695. Though formula (5.70) is formally invalid for small  $n$ , for  $n = 2$ , already we have a 4 percent accuracy of the asymptotic formula (5.70).

If we compare with the experimental situation where the gap to width ratio  $\epsilon \equiv \frac{b}{2a}$  and  $Ca$  are the two convenient control parameters, we have the prediction that for  $1 \gg Ca \gg \epsilon$ ,

$$\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} = 2.776 \frac{\epsilon}{Ca^{3/2}} \quad (5.75)$$

for the  $n = 1$  branch and for  $n = 2$ ,

$$\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} = 4.424 \frac{\epsilon}{Ca^{3/2}} \quad (5.76)$$

Note that for  $1 \gg Ca \gg \mathcal{G}^{1/2}$ , i.e.  $1 \gg Ca \gg \epsilon^{2/3}$ , from formulas (5.71), (5.73) and (5.74), for fixed integer  $n$ ,  $\lambda$  approaches zero like  $\frac{\mathcal{G}^{1/4}}{Ca^{1/2}}$  or  $\frac{\epsilon^{1/2}}{Ca^{3/4}}$ . If on the other hand  $\mathcal{G} \ll Ca \ll \mathcal{G}^{1/2} \ll 1$ , i.e. for  $\epsilon \ll Ca \ll \epsilon^{2/3}$ ,  $\lambda - \frac{1}{2}$  is proportional to  $\frac{Ca}{\mathcal{G}^{1/2}}$  or  $\frac{Ca^{3/2}}{\epsilon}$ .

The most interesting case is when  $Ca = O(\mathcal{G}^{1/2})$ , i.e.  $Ca = \epsilon^{2/3}$  because in that case the selected  $\lambda$  is in the open interval  $(0, \frac{1}{2})$  and the different branches, i.e. different  $n$ , do not asymptote to the same limiting  $\lambda$ . This means that if one of the

solutions were stable and the others unstable, there would be no mechanism for nonlinear stability as put forward by Bensimon.

### 6. Case of $\lambda > \frac{1}{2}$

For  $\lambda > \frac{1}{2}$ , i.e.  $p^2 > 0$ , we will restrict the analysis to the case of  $p \ll 1$ , i.e.  $\lambda$  close to one half. Our primary concern in this section will be to confirm the validity of the previously calculated selection rule (Combescot et al (1986), Shraiman (1986), Hong & Langer (1986), Tanveer (1987b), Dorsey & Martin (1987b) and Dorsey & Martin (1987)) for  $Ca \ll G^{7/3}$ . In this case, it is appropriate to introduce the following transformation of dependent and independent variables:

$$\zeta = \frac{1}{p\xi} \quad (6.1)$$

$$f = p^2 D(\xi) \quad (6.2)$$

$$g = p^2 \omega(\xi) \quad (6.3)$$

Then to the leading order in  $p$ , equations (3.3) and (3.4) reduce to

$$\frac{\xi^2 D'' + \xi D' - 2\xi^2}{(\xi^2 - 1 - \xi D')^{3/2}} - e^{i\pi/6} \frac{\delta_2}{(\xi^2 - 1 - \xi D')^{1/6}} = -i\delta (D + \omega) \quad (6.4)$$

$$\xi \frac{d\omega}{d\xi} = -e^{-i\pi/3} \frac{\delta_3}{(\xi^2 - 1 - \xi D')^{1/3}} + e^{i\pi/6} \delta_4 \frac{(\xi^2 D'' + \xi D' - 2\xi^2)}{(\xi^2 - 1 - \xi D')^{11/6}} \quad (6.5)$$

where

$$\delta = \frac{p^3}{G} \frac{4}{\pi} \quad (6.6)$$

$$\delta_2 = 3.878 \frac{2^{10/3} p^{1/3}}{\pi^2} \sqrt{\frac{\pi^2}{12}} G^{-1/2} Ca^{1/6} \quad (6.7)$$

$$\delta_3 = \frac{1.3375 Ca^{2/3}}{2^{2/3} p^{8/3}} \quad (6.8)$$

$$\delta_4 = 1.3375 \frac{\pi^2}{2^{14/3}} \sqrt{\frac{12}{\pi^2}} G^{1/2} Ca^{7/6} \frac{1}{p^{11/3}} \quad (6.9)$$

Note the differing definition of  $\delta$  from previous work (Tanveer, 1987b). This corresponds to multiplicative adjustment of the parameter  $G$  by  $\frac{\pi}{4}$  as would be necessary for the Mclean-Saffman theory to be valid even for  $Ca \ll G^3$  (assuming zero contact angle between the

advancing interface and the gap plates) as noted earlier. The two equations (6.4) and (6.5) can be conveniently replaced by a single equation by defining a new dependent variable

$$G(\xi) = \xi^2 - 1 - \xi D' \quad (6.10)$$

Then eliminating  $\omega$  between (6.4) and (6.5), we get

$$G'' + \frac{G'}{\xi} - \frac{3}{2} \frac{G'^2}{G} - \delta_2 \frac{e^{i\pi/6}}{3} \frac{G^{1/6}}{\xi} G' = i\delta \frac{G^{3/2}}{\xi^2} [\xi^2 - 1 - G] - i\delta\delta_3 e^{-i\pi/3} \frac{G^{7/6}}{\xi^2} - i\delta\delta_4 \frac{e^{i\pi/6}}{\xi} \frac{G'}{G^{1/3}} \quad (6.11)$$

One needs to look for solutions to (6.11) for different possibilities of the parameters  $\delta_2$ ,  $\delta_3$ ,  $\delta_4$  and  $\delta$  such that they match with the outer solution in sectors I, II and III of Fig. 3. Depending on the relative size of these parameters, there is need for a set of nested inner regions as in section 5. In this paper, we do not address the possibility of solutions for different ranges of parameter. We only concentrate on the specific case  $Ca \ll \mathcal{G}^{7/3}$  and  $\delta = O(1)$ . It is clear that in this case  $\delta_2$ ,  $\delta_3$  and  $\delta_4$  are each much smaller than unity so that the thin film effect totally drops out. In that case, in order to relate to earlier work (Tanveer, 1987b), it is convenient to go back to the original equation (6.4) and (6.5) which is now reduced to one equation

$$\frac{\xi^2 D'' + \xi D' - 2\xi^2}{(\xi^2 - 1 - \xi D')^{3/2}} = -i\delta D \quad (6.12)$$

which is the same nonlinear equation as (Tanveer(1987b)) (equation 127 of that paper when  $\delta_1 = \delta$  and  $\alpha = 1$ , i.e. symmetry is assumed) as for the MS theory. From previous work, one needs to find solutions of (6.12) so that

$$D \sim \frac{-2i}{\delta\xi} \quad (6.13)$$

for argument  $\xi$  in the interval  $(-\pi, 0)$  in order to match with the regular perturbation series in  $\mathcal{G}$ . For symmetric fingers, it is enough to require that the asymptotic behavior

(6.12) hold for large  $\xi$  with  $\text{Arg } \xi$  in the interval  $[-\pi/2, 0)$  with the requirement that the solution be completely real on the negative imaginary  $\xi$  axis for sufficiently large  $\xi$ . This condition determined  $\delta$  when it was order unity. For large  $\delta$  asymptotic analysis is possible and the details are given in the previous work (Tanveer, 1987b) which is equivalent to earlier analysis of the MS equations by Combescot et al<sup>7</sup>.

When  $\delta = O(1)$  and  $Ca = O(\mathcal{G}^{7/3})$ , the  $\delta_2$  factor is order unity, whereas each of  $\delta_3$  and  $\delta_4$  is negligible compared to unity. In that case, we can still reduce to one equation

$$\frac{\xi^2 D'' + \xi D' - 2\xi^2}{(\xi^2 - 1 - \xi D')^{3/2}} - e^{i\pi/6} \frac{\delta_2}{(\xi^2 - 1 - \xi D')^{1/6}} = -i\delta D \quad (6.13)$$

and numerical computation of these solutions using the same procedure as detailed in the earlier paper (Tanveer, 1987b) shows that for  $\delta_2 = O(1)$ , as we increase  $\delta_2$  from 0 to 10, the corresponding  $\delta$  increases monotonically. This means that the finger is fatter than for the MS boundary conditions when parameters are such that the variation of transverse curvature is more important than the thin film leakage term. This is consistent with direct numerical computation of Schwartz & DeGregoria (1987) where inclusion of just the transverse variation of curvature term while neglecting flow into the thinfilm region resulted in fingers with width more than  $\frac{1}{2}$  in the limit of small  $\mathcal{G}$ .

We leave the detailed analysis for other ranges of  $Ca$  and  $\mathcal{G}$  for  $\lambda > \frac{1}{2}$  for the future.

## 7. Analytic theory for arbitrary capillary number:

Eliminating  $g$  between (3.3) and (3.4) one finds

$$\begin{aligned} \zeta f' = & \zeta h'_1 - \sqrt{\frac{4\mathcal{G}Ca}{3}} \kappa^{0'}(Ca n_x) n'_x + \zeta h'_2 - \frac{4\mathcal{G}}{\pi} \left[ \frac{1}{R}(\zeta) \right]' \kappa^1(Ca n_x) \\ & - \frac{4\mathcal{G}}{\pi} \frac{1}{R} \kappa^{1'}(Ca n_x) - h_3(\zeta) + \frac{l_2 - l_1}{\zeta^2 - 1} m^0(Ca n_x) - h_4(\zeta) \\ & + \sqrt{\frac{12\mathcal{G}Ca}{\pi^2}} \frac{1}{R} \left( \frac{l_2 - l_1}{\zeta^2 - 1} \right) m^1(Ca n_x) + \alpha \end{aligned} \quad (7.1)$$

where  $n_x$ ,  $\frac{1}{R}$ ,  $l_1$  and  $l_2$  are as in (2.18), (2.19), (2.20) and (2.21) respectively;  $h_1$ ,  $h_2$ ,  $h_3$  and  $h_4$  are as defined in (3.5)-(3.8). Note that on taking the derivative of (2.18) with respect to  $\zeta$

$$n'_x = \frac{i}{2} \left[ \frac{l_2^{-1/2} l'_2}{l_1^{1/2}} - \frac{l_2^{1/2} l'_1}{l_1^{3/2}} - \frac{l_1^{-1/2} l'_1}{l_2^{1/2}} + \frac{l_1^{1/2} l'_2}{l_2^{3/2}} \right] \quad (7.2)$$

and this involves the second derivative of  $f$

For  $\mathcal{G} = 0$ ,

$$\zeta f'_0 + h_{3_0}(\zeta) - \alpha^0 - m^0(Ca n_x^0) \frac{l_{2_0} - l_{1_0}}{\zeta^2 - 1} = 0 \quad (7.3)$$

as can be obtained from (7.1) or through direct analytic continuation of (2.23) where

$$h_{3_0}(\zeta) \equiv \mathcal{I} \left( m^0(Ca n_x^0) \frac{1}{2} \frac{l_{2_0} - l_{1_0}}{\zeta^2 - 1} \right) \quad (7.4)$$

$$l_{2_0} = 1 - p^2 \zeta^2 + \zeta (\zeta^2 - 1) f'_0 \quad (7.5)$$

$$l_{1_0} = \zeta^2 - p^2 + \frac{1}{\zeta} (1 - \zeta^2) f'_0 \left( \frac{1}{\zeta} \right) \quad (7.6)$$

Equation (7.3) has no exact solution. We can determine  $f_0$  for  $|\zeta| \leq 1$  using the convenient power series representation in (2.24). However, we are interested in the behavior of  $f_0$  and the next order regular perturbation term  $f_1$  for  $|\zeta| > 1$  to find possible sources of nonuniformity of the expansion. If we define

$$f'_K(\zeta) = f'_0(\zeta^{-1}) \quad (7.7)$$

then numerical calculation of the coefficients in (2.24) allows us to calculate  $f'_K$  and its derivative (which is related to  $f''_0(1/\zeta)$ ) conveniently. Similarly the power series in (2.23) allows us to calculate  $h_{3_0}$ . Equation (7.3) then becomes a nonlinear algebraic equation to determine  $f'_0(\zeta)$  for  $|\zeta| > 1$ , which can be determined numerically by Newton iteration.

Note that this kind of trick can be of wide applicability in other problems such as Kelvin-Helmholtz and Rayleigh-Taylor problems where use of numerical calculations and

conversion of nonlocal equations into local algebraic and differential equations can be exploited to study the evolution of singularities in the unphysical plane using a numerical procedure.

Assuming  $f'_K$ ,  $f''_K$  and  $h_{3_0}$  as given, the nonlinear algebraic equation (7.3) determining  $f'_0$  for any given  $\zeta$  outside the unit circle, can be symbolically expressed as

$$E(f'_0, \zeta) = 0 \quad (7.8)$$

where  $E$  is a function of two independent variables. From implicit function theorem, (7.8) determines  $f'_0(\zeta)$  such that  $E(f'_0(\zeta), \zeta) = 0$ .

From equation (7.1), one can easily see that the next order perturbation term  $f_1$  as expressed in (2.25) satisfies an equation that can be written symbolically as

$$E_{f'_0}(f'_0(\zeta), \zeta) f'_1(\zeta) = \mathcal{R}_1 + \mathcal{R}_2 f''_0 \quad (7.9)$$

The expressions for  $\mathcal{R}_1$  and  $\mathcal{R}_2$  can be obtained from (7.3) by a straightforward though lengthy algebra. Note that each of  $\mathcal{R}_1$  and  $\mathcal{R}_2$  involve  $f'_0(\zeta)$  besides  $f'_K$ ,  $f''_K$ ,  $f'_1(\zeta^{-1})$  and  $h_{3_0}$ . Each of the latter four functions are considered known as they are readily determined by using the power series representations for  $f_0$  and  $f_1$  as  $|\zeta^{-1}| < 1$ . It is clear now that there is a singularity of  $f'_1$  at  $\zeta = \zeta_0$  where

$$E_{f'_0}(f'_0(\zeta_0), \zeta_0) = 0 \quad (7.10)$$

It is not difficult to see that at  $\zeta = \zeta_0$ , where  $E_{f'_0}(f'_0(\zeta), \zeta) = 0$ , the second derivative  $f''_0$  is also singular. If we define

$$A = f'_0(\zeta_0) \quad (7.11)$$

$$A_1 = \mathcal{R}_2(f'_0(\zeta_0), \zeta_0) \quad (7.12)$$

$$A_2 = E_{\zeta}(f'_0(\zeta_0), \zeta_0) \quad (7.13)$$

$$A_3 = -\frac{1}{2} E_{f'_0, f'_0}(f'_0(\zeta_0), \zeta_0) \quad (7.14)$$

Then it is not difficult to see that in the neighborhood of  $\zeta = \zeta_0$

$$f'_0 \sim A - \sqrt{\frac{A_2}{A_3}} (\zeta - \zeta_0)^{1/2} \quad (7.15)$$

Near  $\zeta = \zeta_0$ , which is a point of nonuniformity of the outer perturbation expansion, it is appropriate to introduce inner variables

$$\zeta - \zeta_0 = u_1 \xi \quad (7.16)$$

$$f' = f'_0(\zeta_0) + u_2 F'(\xi) \quad (7.17)$$

where

$$u_1 = \frac{A_1^{2/3} \mathcal{G}^{1/3}}{A_3^{1/3} A_2^{1/3}} \quad (7.18)$$

and

$$u_2 = \frac{A_1^{1/3} A_2^{1/3} \mathcal{G}^{1/6}}{A_3^{2/3}} \quad (7.19)$$

Then for  $\xi = O(1)$ , equation (7.1) to the leading order in  $\mathcal{G}$  reduces

$$-F'^2 + \xi = F'' \quad (7.20)$$

which is exactly the same equation as in section 5c for  $\xi_2 = O(1)$ . Note that the highest derivative term  $F'''$  does not appear at the leading order. However the appropriate solution to (7.20) is

$$F' = \frac{Ai'(\xi)}{Ai(\xi)} \quad (7.21)$$

which is singular at the zeroes of  $Ai(\xi)$  and there exists an inner neighborhood around each of the zeroes that where the third derivative term will be important.

Note in order that the behavior of  $F'$  be given by the specific solution (7.21) to (7.20), the structure of the Stokes lines should be qualitatively the same as that found in case (5c). This is expected to be true when  $Ca$  is less than 0.2 or so, i.e. within the experimental range; however this needs to be verified numerically.

The solution (7.21) is matched to the outer solution

$$f' \sim f'_0 + \mathcal{G}^{1/2} f'_1 + ..C_1 e^{g^{-1/2} w_0^+ + w_1^+} \quad (7.22)$$

for large  $\xi$  with corresponding  $\zeta$  on the imaginary axis and by following the procedure in section 5c, it is not difficult to see that the condition of  $C_1$  being real is equivalent to

$$\begin{aligned} -\mathcal{G}^{-1/2} \text{Im} \left[ W_0(\zeta_I) - W_0(\zeta_0) + \mathcal{G}^{1/2} (W_1(\zeta_I) - W_0(\zeta_0)) \right] - \pi/2 + \text{Arg } u_2 \\ - \frac{1}{2} \text{Arg } u_1 = n \pi \end{aligned} \quad (7.23)$$

where  $\zeta_I$  is any point on the imaginary  $\zeta$  axis just beyond  $\zeta = i$ , and  $n$  is some integer.

Equation (7.23) is the selection rule. In order to get concrete results one needs  $m^0$ ,  $m^1$ ,  $\kappa^0$  and  $\kappa^1$  for complex arguments since the analysis in section (5c) suggests that  $\zeta_0$  will not be on the imaginary  $\zeta$  axis, where  $n_*$  is real, but somewhat off the axis. From calculation, we find

$$E_{f'_0} (f'_0(\zeta), \zeta) = \zeta (1 + Q_{3_0}) \quad (7.24)$$

where

$$Q_{3_0} = -m^0(Ca n_*^0(\zeta)) - Ca \frac{i}{4} \frac{(l_{2_0} + l_{1_0})}{l_{2_0}^{3/2} l_{1_0}^{1/2}} (l_{2_0} - l_{1_0}) m^{0'}(Ca n_*^0) \quad (7.25)$$

Thus, the singularity point is where  $Q_{3_0} = -1$ . For small  $Ca$ , noting the behavior of the Reinel functions, this can only happen close to where  $l_{2_0}^{-1}$  is large, i.e. near  $\zeta = \frac{i}{4}$ , the zero of the derivative of the conformal map corresponding to the Saffman-Taylor solution.

The effect of finite  $Ca$  would perhaps be to move  $\zeta_0$  even further from the imaginary axis. However, there is an important issue that we are unable to answer at this point. If  $\zeta_0$  moves way off the imaginary  $\zeta$  axis, the determination of finger width selection involves knowledge of the analytic functions  $m^0, \dots$ , etc for complex arguments whose imaginary part can be large. Numerically, one only finds approximate solutions for  $m^0, \dots$  for real

values of the arguments. There could two analytic functions that are quite close to each other on the real axis but differ significantly as we move off the real axis. If the problem is well posed for small  $\mathcal{G}$ , the deviations of the two functions just mentioned in the complex plane should not matter. However, if  $\zeta_0$  has a significantly large real part, it will matter. Does this mean that  $\zeta_0$  cannot move too far from the imaginary axis? Or does it mean that the problem is ill posed for even non zero  $\mathcal{G} \ll 1$  for  $Ca = O(1)$  in the sense that insignificant changes in the form of the functions  $m^0$ ,  $m^1$ ,  $\kappa^0$  and  $\kappa^1$  on parts of the real physical domain cause large changes in the selection. One is left to wonder if this has anything to do with the evolution problem in the time dependent case where fractal like structures (Maxworthy, 1987) have been observed in experiments at sufficiently large  $Ca$ .

### 8. Summary of theoretical results and comparison with experimental data:

In this paper, we have found concrete results only in the limit where both  $\mathcal{G}$  and  $Ca$  are small, i.e.  $\epsilon^2 \ll Ca \ll 1$ , where  $\epsilon$  is the gap to width ratio  $\frac{b}{2a}$ .

For  $\lambda > \frac{1}{2}$ , we have found that the finger solutions of the MS theory where  $\lambda \sim \frac{1}{2} + \text{constant } \mathcal{G}^{2/3}$  persist for  $Ca \ll \mathcal{G}^{7/3} \ll 1$  provided we make a multiplicative adjustment of the previously calculated constant (Tanveer, 1987b, Dorsey & Martin, 1987) by a factor of  $(\frac{\pi}{4})^{2/3}$ . The possibility of other solutions with  $\lambda > \frac{1}{2}$  for other ranges in the parameter space has not been investigated.

For  $\lambda < \frac{1}{2}$ , for  $\mathcal{G} \ll Ca \ll 1$ , we predict a discrete set of finger solutions for which

$$\frac{\lambda^2 (1 - \lambda)}{(1 - 2\lambda)} = k_n \frac{\mathcal{G}^{1/2}}{Ca} \quad (8.1)$$

where

$$k_1 = 3.061 \quad (8.2)$$

$$k_2 = 4.878 \quad (8.3)$$

and for  $n \gg 1$ ,

$$k_n \sim 2.06 (n + 0.279) \quad (8.3)$$

Unfortunately, for most of the experiments to date, strict quantitative test for the theoretical result (8.1) is not possible because most of the experimental data shows that  $\epsilon$  is not small enough to make  $Ca$  and  $\mathcal{G}$  small at the same time (note  $\mathcal{G} = \epsilon^2 \pi^2 / (12Ca)$ ). However, for the Tabeling-Zocchi-Libchaber experiment, the data shows general qualitative agreement with (8.1) with  $n = 1$ , i.e. the first branch, as shown in table 1 for some of the smallest values of  $\mathcal{G}$  available. The theoretically predicted  $\lambda$  in table I was obtained by numerically solving the nonlinear equation (8.1) using standard Newton iteration. For the range of the Tabeling et al experiment, we do not have any data for which the right hand side of (8.1) is small compared to unity. Note that for small but fixed values of  $\mathcal{G}$  as  $\epsilon$  is decreased, both the theory and experiment suggest that  $\lambda$  approaches  $1/2$ . From (8.1), we get  $1 - 2\lambda$  proportional to  $\epsilon^2 / \mathcal{G}^{3/2}$  for  $\mathcal{G} \ll \epsilon \ll \mathcal{G}^{3/4} \ll 1$ .

Table 2 shows that there is very little agreement with the Kopf-Sill & Homsy experiment where very skinny fingers were observed. It appears that in their experiment, the SPHR conditions will be invalid in some small neighborhood near the tip, because the product of the gap width  $b$  and the tip curvature (as calculated from the Saffman-Taylor theoretical formula) is order unity. It is clear that in the Reinelt(1987a) derivation of the interfacial boundary conditions, one assumes that the product of lateral curvature and the gap width is small everywhere. However, this product at points away from the tip is small for their experimentally observed finger and so the SPHR conditions hold except right near the tip. We suggest that this in some sense is equivalent to a tip perturbation on the regular Saffman-Taylor finger and that might explain the similarity of the observed features with that of Couder et al (1986) where a small bubble near the tip of the finger is found to dramatically affect the selection mechanism. However, we are not sure how to explain the observation of Kopf-Sill & Homsy that the skinny fingers are only found for extremely clean systems. The theory presented here assumes the SPHR conditions to hold everywhere on the interface and is therefore unable to account for the experiment. Note that in our theoretical prediction, when  $\mathcal{G}^{1/2} \ll Ca \ll 1$ , the width  $\lambda$  is

small. However, the product of tip curvature and the gap width, which is proportional to  $\lambda^{-2} \epsilon$  for small  $\lambda$  (using Saffman- Taylor formula), is proportional to  $Ca^{3/2}$  and is far smaller than unity as required for the validity of the SPHR conditions everywhere on the finger boundary.

We conclude this section by discussing a finding of some theoretical importance. If we go back to the analysis of section 5c, we notice that it is necessary to carry out matching in a sequence of nested inner regions,  $\xi_3 = O(1)$ ,  $\xi_1 = O(1)$  and then  $\xi_2 = O(1)$ . Note that the leading order equations in each of the inner regions contain parameters. If at the outset, we had linearized the equations about  $F = 0$ , these set of nested regions would not exist. Indeed, it is the nonlinearity of  $F_{1_0}$  that generated a nonuniformity in  $F_{1_1}$ , which accounted for the inner region where  $\xi_2 = O(1)$ . The matching of the behavior of this region to the next outer region where  $\xi_1 = O(1)$  is what determined the selection rule (8.1). A linearized equation would have failed to predict even the correct scaling. It may be pointed out that the linearized analysis as of Shraiman (1986) has been extensively used in selection problems both for the Saffman-Taylor problem as well as in dendritic crystal growth. For the Saffman-Taylor problem with the MS boundary conditions, the results are quite close to what one finds from a fully nonlinear analysis (Combescot et al (1986), Tanveer (1987b), Dorsey & Martin (1987)), the only difference being a small error in values of scaling constants. The reason that the linearized analysis works for the MS boundary conditions is that in the inner region, where the nonlinear analysis is relevant, there is only one parameter  $\frac{\lambda - \frac{1}{2}}{\sigma^{2/3}}$  in the nonlinear equation that has to be determined by matching to the outer solution. If one replaced this nonlinear equation by a linear one, there will still be the same one parameter in the problem though the actual numerical value of that parameter will not be correct. However, this parameter contains all the scaling information. This is obviously not true in our problem with the thin film and the nonlinear analysis is absolutely essential to get the correct scaling laws as well as numerical constants.

We have presented an analytic theory for the selection of Saffman-Taylor finger in the presence of the thin film that is neglected in the previous theory based on the Mclean-Saffman boundary conditions. Precise scaling laws are calculated. There is prediction of very skinny fingers. Unfortunately, direct quantitative verification from existing experiments is not possible because Bretherton's results are only valid for rather small  $Ca$ , and therefore a very small gap width to cell width ratio, far smaller than the existing experiments, is necessary in order that  $G$  be small enough for the validity of our scaling laws. However, there is qualitative agreement with one set of experiments. The completion of details for arbitrary  $Ca$  provides some exciting possibilities that will be left for the future. The importance of nonlinear analysis has been pointed out. The linear stability of the steady states for boundary conditions incorporating thin film effects is also an important problem that is left for the future.

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| $\epsilon$ | $Ca$   | $1/B$ | $\mathcal{G}$          | $3.061 \frac{\mathcal{G}^{1/2}}{Ca}$ | Expt. $\lambda$ | Theor. $\lambda$ |
|------------|--------|-------|------------------------|--------------------------------------|-----------------|------------------|
| 1/112.5    | 0.0395 | 6000  | $0.165 \times 10^{-2}$ | 3.14                                 | 0.48            | 0.481            |
| 1/65       | 0.118  | 6000  | $0.165 \times 10^{-2}$ | 1.05                                 | 0.47            | 0.447            |
| 1/112.5    | 0.0263 | 4000  | $0.247 \times 10^{-2}$ | 5.77                                 | 0.49            | 0.489            |
| 1/65       | 0.0789 | 4000  | $0.247 \times 10^{-2}$ | 1.93                                 | 0.48            | 0.470            |

Table 1. Comparison with Tabeling et al (1987) experiment for a few values of  $\epsilon$  and  $Ca$  for which  $\mathcal{G}$  was small. Note that  $B = \mathcal{G}/\pi^2$ , and is the same  $B$  introduced by Tabeling et al. Also, note that values of experimental  $\lambda$  quoted here are eyeball readings from Fig. 8 of the Tabeling et al (1987) paper.

| $\epsilon$            | $Ca$   | $\mathcal{G}$         | $3.061 \frac{\mathcal{G}^{1/2}}{Ca}$ | Expt. $\lambda$ | Theor. $\lambda$ |
|-----------------------|--------|-----------------------|--------------------------------------|-----------------|------------------|
| $0.38 \times 10^{-2}$ | 0.25   | $0.49 \times 10^{-4}$ | 0.089                                | 0.15            | 0.24             |
| $0.38 \times 10^{-2}$ | 0.054  | $0.22 \times 10^{-3}$ | 0.85                                 | 0.20            | 0.44             |
| $0.38 \times 10^{-2}$ | 0.0128 | $0.95 \times 10^{-3}$ | 7.35                                 | 0.25            | 0.49             |
| $0.38 \times 10^{-2}$ | 0.0036 | $0.33 \times 10^{-2}$ | 48.44                                | 0.31            | 0.50             |

Table 2. Comparison with Kopf-Sill & Homsy (1987) experiment.

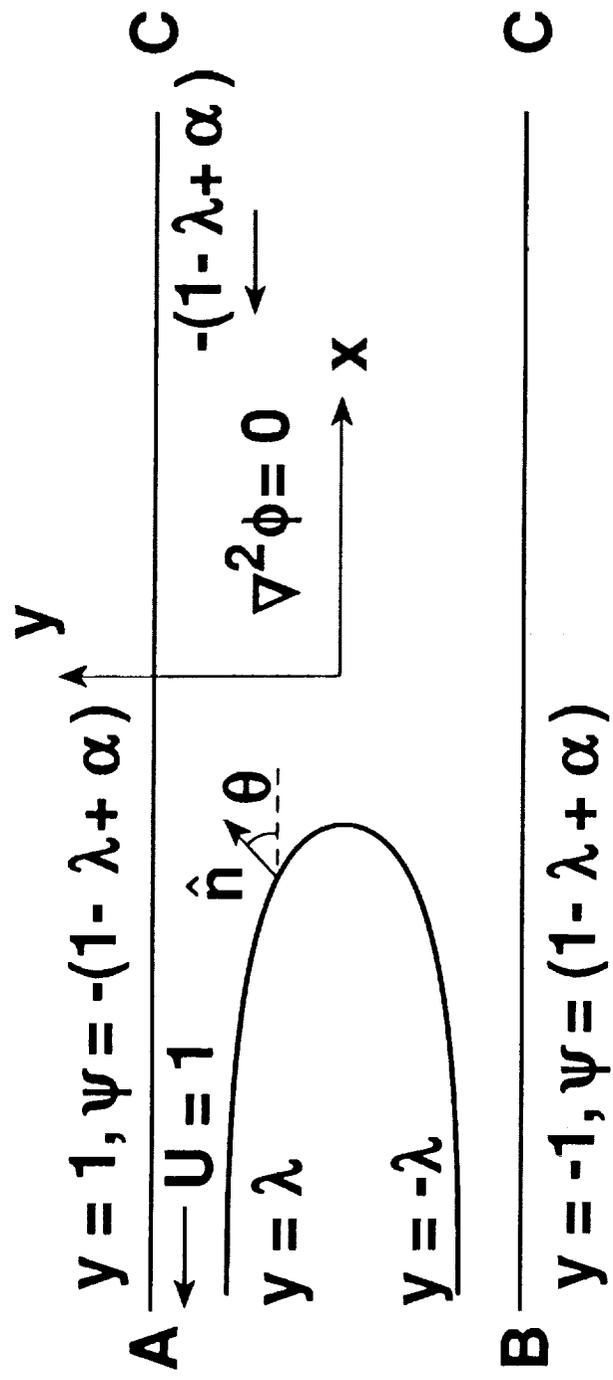


Figure 1.

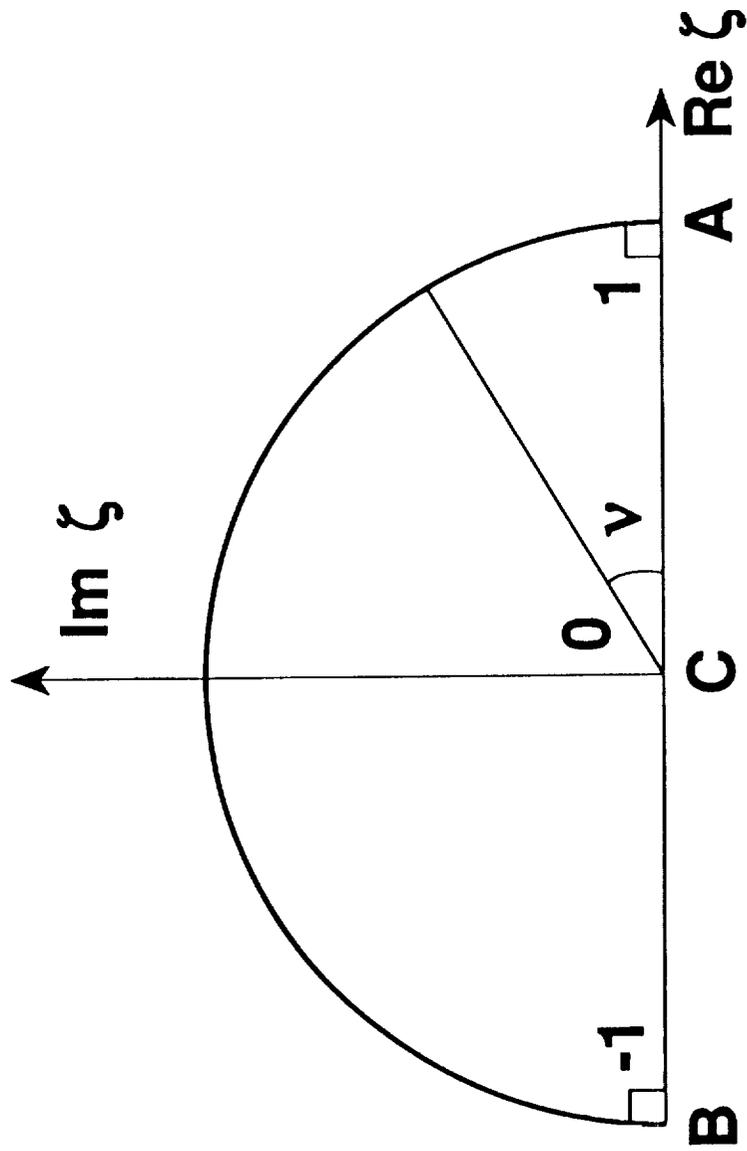


Figure 2.

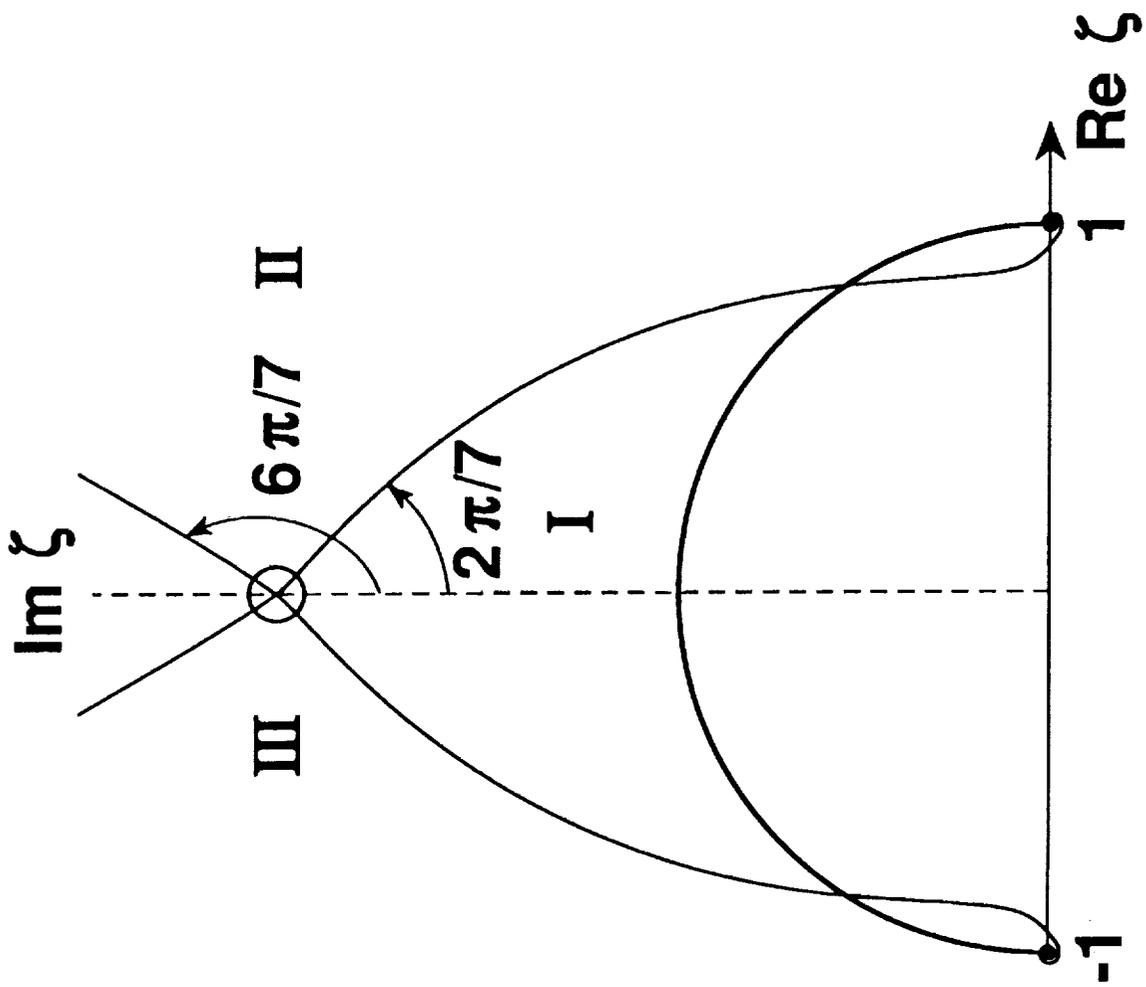


Figure 3.

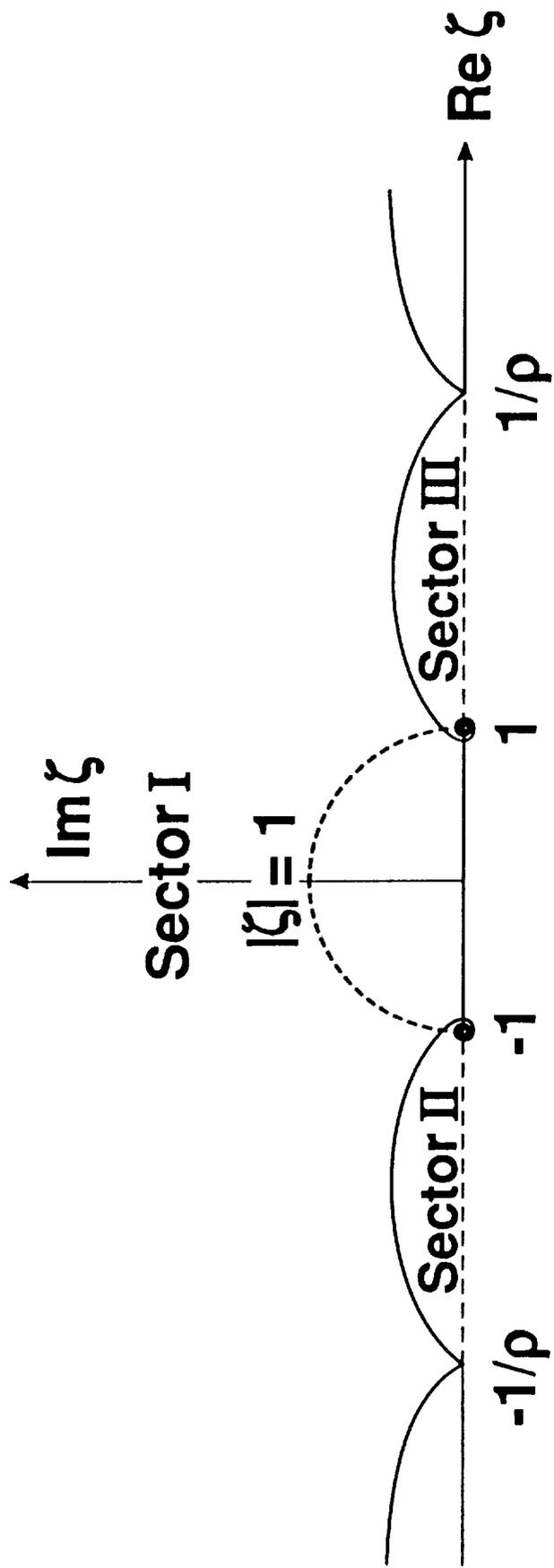


Figure 4.

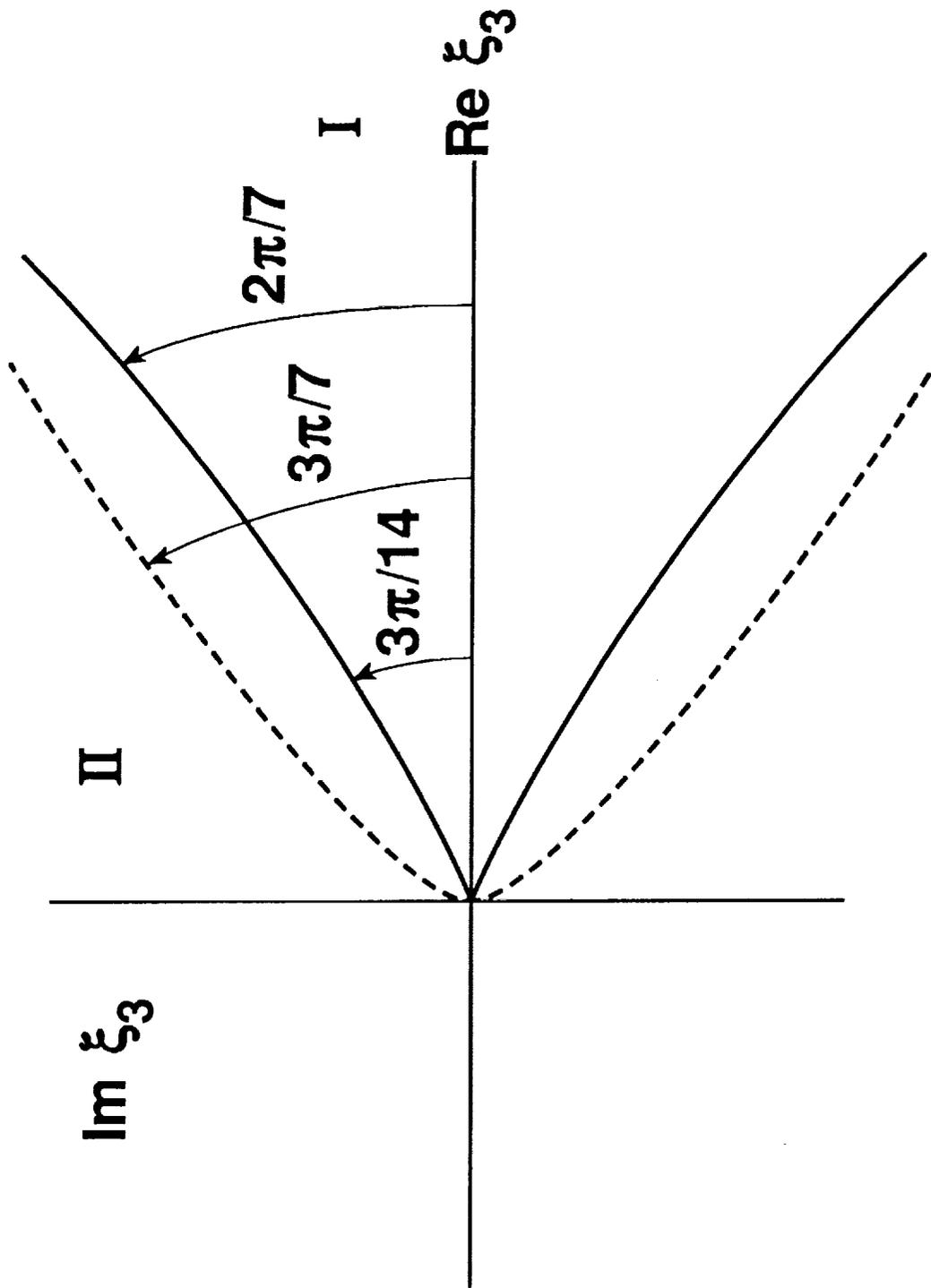


Figure 5.

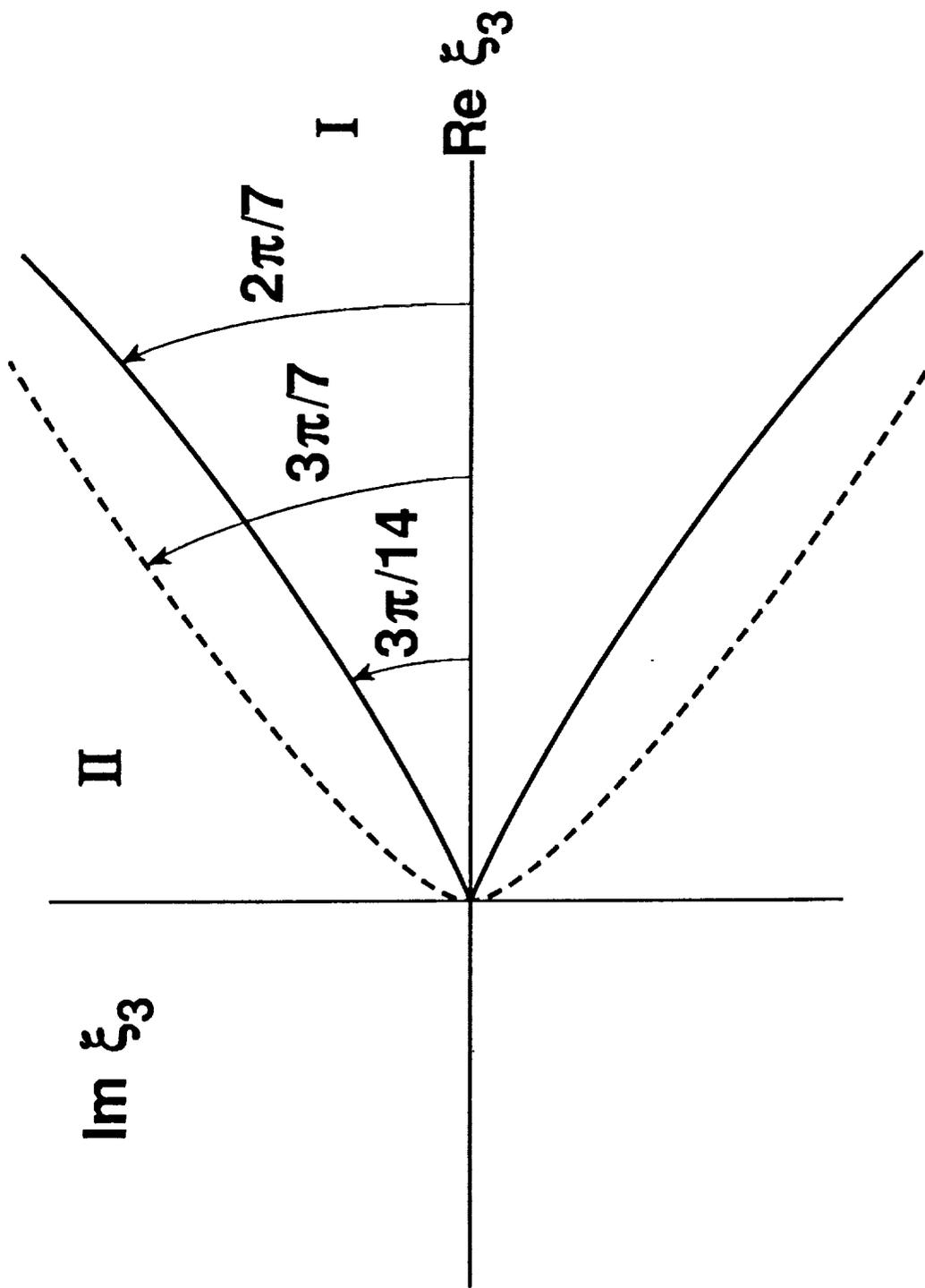


Figure 6.





# Report Documentation Page

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| 16. Abstract<br><br>An analytic theory is presented for the width selection of Saffman-Taylor fingers in the presence of thin film effect. In the limit of small capillary number $Ca$ and small gap to width ratio $\epsilon$ , such that $\epsilon \ll Ca \ll 1$ , it is found that fingers with relative width $\lambda < \frac{1}{2}$ are possible such that $\frac{\lambda^2(1-\lambda)}{(1-2\lambda)} = k \frac{\epsilon}{Ca^{3/2}}$ , where the positive constant $k$ depends on the branch of solution and equals 2.776 for the first branch. A fully nonlinear analysis is necessary in this problem even to obtain the correct scaling law. It is also shown how in principle, the selection rule for arbitrary $Ca$ can be obtained. |  |   |  |
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